SGM-manifolds

The following properties of compact (M,g) can be expressed in terms of $d : \partial M \times \partial M \to \mathbb{R}$ only:

① All geodesic segments strongly minimize:

$\forall \xi \in B\partial M = \{\xi \in T\partial M \mid |\xi|_g < 1\}$ $\exists!\, y \in \partial M$ s.t.

(i) $\operatorname{grad}_{\partial M} d(y,\cdot)|_x = \xi$, $x = \pi(\xi)$

(ii) $\nexists\, z \in \partial M \setminus \{x,y\} : d(x,y) = d(x,z) + d(z,y)$.

$\boxed{\text{Assume ①}}$

② $\gamma : [a,b] \to \partial M$ is a geodesic of M (as well as of ∂M):

$L(\gamma) = d(x,z)$, $\gamma(a) = x$, $\gamma(b) = z$

$\exists \{\xi_i\} \subset B_x\partial M$, $\xi_i \to \dot\gamma(a)$ s.t. $\{y_i\}$ determined from ① converge to $y \in \partial M$ s.t. $d(x,z) + d(z,y) = d(x,y)$

Then we say that γ is a <u>straight segment</u> of ∂M.

③ The length minimizing path $\gamma : [a,b] \to M$ from x to y is a (possibly grazing) geodesic:

$\{z \in \partial M \mid d(x,z) + d(z,y) = d(x,y)\}$ is a countable union of straight segments and points.

Then we say that x & y are <u>straight connected</u>.

④ There do not exist infinite grazing geodesics:

$\nexists \{x_i\} \subset \partial M$ s.t. x_i & x_{i+1} are straight connected,

$\|\operatorname{grad}_{\partial M}(x_{i\pm 1},\cdot)|_i\| = 1$, $\operatorname{grad}_{\partial M}(x_{i-1},\cdot)|_i = -\operatorname{grad}_{\partial M}(x_{i+1},\cdot)|_i$

and $\sum_{i=0}^{\infty} d(x_i, x_{i+1}) = \infty$.

<u>Definition:</u> (M,g) is SGM, if it satisfies ① & ④.

Lemma: If (M,g) and $(\tilde M,\tilde g)$ are SGM and $d=\tilde d$, then $F=\tilde F$, where $F:\partial_-SM \to \mathbb{R}\times\partial SM$; $F(\xi)=(\tau_+(\xi), \Phi_H^{\tau_+(\xi)}(\xi))$.

Proof: $d=\tilde d \Rightarrow (\partial M, g|_{\partial M}) = (\partial\tilde M, \tilde g|_{\partial \tilde M})$.

Let $\xi_0\in\partial_-SM$ and $\xi\in B\partial M$ s.t. $\xi_0=\xi+\sqrt{1-|\xi|^2}\,\nu_-$.

Let $y\in\partial M$ be determined by ①.

Then y is the first point where the geod. emanating ~~from~~ ξ_0 hits to ∂M, and $\tau_+(\xi_0)=d(x,y)$, because it must be a length minimizing curve. $\pi(\xi_0)$

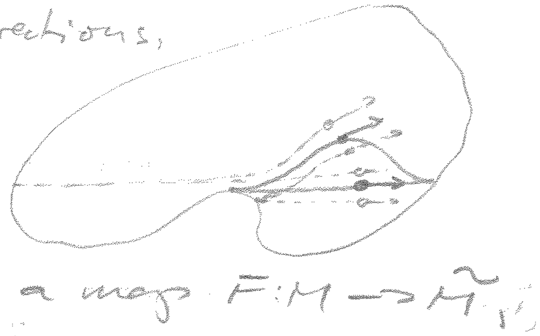
Now the points y coincide for (M,g) & $(\tilde M,\tilde g)$ since they are determined by $d=\tilde d$ only!

Now there exists a unique $\eta_0\in\partial_-SM$ at y which gives rise to a geodesic which hits ∂M at x. Furthermore $F(\xi_0)=(d(x,y),\eta_0)$ and η_0 is also determined by $d=\tilde d$ only, so $F=\tilde F$ □

Lemma: Let (M,g) and $(\tilde M,\tilde g)$ be SGM and $d=\tilde d$.

Then $\mathcal{G}: SM\to S\tilde M$; $\mathcal{G}(\xi)=\Phi_{\tilde H}^{\tilde\tau_-(\xi)}\circ\Phi_H^{-\tau_-(\xi)}(\xi)$ is a homeomorphism and $\mathrm{Vol}(M)=\mathrm{Vol}(\tilde M)$.

Proof: $\mathcal{G}$ is diffeo except at tangential directions, where it is local homeo.
$\mathrm{Vol}(M)=\mathrm{Vol}(\tilde M)$ follows from Santaló-formula



Lemma: If we can show that $\mathcal{G}$ covers a map $F:M\to\tilde M$, then F must be an isometry.

Proof: Assume that $\exists\xi\in\partial_-SM:\begin{cases}x=\Phi_H^t(\xi)\\ y=\Phi_H^s(\xi)\end{cases}$. Then $\begin{cases}F(x)=\Phi_{\tilde H}^t(\xi)\\ F(y)=\Phi_{\tilde H}^s(\xi)\end{cases}$

and since all geodesics are minimizing,

$$d_M(x,y)=d_{\tilde M}(F(x),F(y))$$

□

Theorem: Let (M,g) be SGM. If $\tilde{g} = \Omega^2 g$ and $\tilde{d}_{\partial M} = d_{\partial M}$, then $\Omega = 1$

Proof:

$$|S^{n-1}|\int_M \Omega(x)\,dx = \int_{SM}(\Omega\circ\pi)(\xi)\,d\xi = \int_{SM}|\xi|_{\tilde{g}}\,d\xi$$

$$\left[\,|\xi|_{\tilde{g}} = \sqrt{\tilde{g}(\xi,\xi)} = \Omega(\pi(\xi))\underbrace{\sqrt{g(\xi,\xi)}}_{=1 \text{ on } SM} = (\Omega\circ\pi)(\xi)\,\right]$$

$$\underset{(\text{Santaló})}{=} \int_{\partial_- SM}\int_0^{\tilde{\tau}_+(\eta)} \underbrace{|\Phi_H^t(\eta)|_{\tilde{g}}}_{=|\dot{\gamma}(t)|_{\tilde{g}}}\,dt\; g(\nu_-,\eta)\,d\eta$$

where $\gamma(t) = \pi\circ\Phi_H^t(\eta)$.

$$\left[\begin{array}{l}\text{Since } \cancel{\text{boundary}} \text{ geodesics minimize on } (M,\tilde{g}), \\ \displaystyle\int_0^{\tilde{\tau}_+(\eta)} |\Phi_H^t(\eta)|_{\tilde{g}}\,dt = \tilde{L}(\gamma) \geq \tilde{\tau}_+(\eta)\end{array}\right]$$

$$\geq \int_{\partial_- SM}\tilde{\tau}_+(\eta)\,g(\nu_-,\eta)\,d\eta = \int_{SM}d\xi = |S^{n-1}|\operatorname{Vol}_g(M)$$

$$\Longrightarrow \boxed{\int_M \Omega(x)\,dx \geq \operatorname{Vol}_g(M)}$$

$$\int_M \Omega(x)\,dx \underset{\text{eq only if } \Omega(x) \text{ constant}}{\leq} \left(\int_M \Omega(x)^n dx\right)^{\frac{1}{n}}\left(\int_M dx\right)^{\frac{n-1}{n}} = \operatorname{Vol}_{\tilde{g}}(M)^{\frac{1}{n}}\operatorname{Vol}_g(M)^{\frac{n-1}{n}}$$

$$\Rightarrow \operatorname{Vol}_g(M) \leq \operatorname{Vol}_{\tilde{g}}(M)^{\frac{1}{n}}\operatorname{Vol}_g(M)^{\frac{n-1}{n}} \Rightarrow \operatorname{Vol}_g(M) \leq \operatorname{Vol}_{\tilde{g}}(M)$$

But g & $\tilde{g}$ can be interchanged to get eq, and hence $\Omega = c = 1$

$\square$

Sasaki-metric on TM

(M,g) Riemannian $\Rightarrow$ $\gamma: [a,b] \to M$ is a geodesic iff

$\dot\gamma: [a,b] \to TM$ is an integral curve of $H_\xi = \xi^i \frac{\partial}{\partial x^i}\big|_\xi - \xi^i \xi^j \Gamma^k_{ij} \frac{\partial}{\partial \xi^k}\big|_\xi$.

Define $\frac{\delta}{\delta x^i}\big|_\xi := \frac{\partial}{\partial x^i}\big|_\xi - \xi^j \Gamma^k_{ij} \frac{\partial}{\partial \xi^k}\big|_\xi$. $\quad (\Rightarrow H_\xi = \xi^k \frac{\delta}{\delta x^k})$

Then $\frac{\delta}{\delta x^1}\big|_\xi, \dots, \frac{\delta}{\delta x^n}\big|_\xi$ form a basis for a subspace $H_\xi TM \subset T_\xi TM$

complementary to $V_\xi TM := \mathrm{Ker}(\pi_*|_\xi)$, $\quad \pi: TM \to M$

$\pi_*|_\xi : T_\xi TM \to T_x M$

Now $\pi_*|_\xi : H_\xi TM \to T_x M$ is a linear isomorphism as well as

$K|_\xi : V_\xi TM \to T_x M \;;\; K|_\xi\left(X^i \frac{\delta}{\delta x^i}\big|_\xi + V^i \frac{\partial}{\partial \xi^i}\big|_\xi\right) = V^i \frac{\partial}{\partial x^i}\big|_{\pi\xi}$

and we can make $H_\xi TM$ & $V_\xi TM$ orthogonal by defining

$$\hat g(X, Y) := g(\pi_*(X), \pi_*(Y)) + g(K(X), K(Y)).$$

In this metric the volume form μ_{SM} of SM is invariant under the geodesic flow $\Phi_H: \mathbb{R} \times SM \to SM$!

Q: What is the normal vector of ∂SM ?

A: $\partial SM = \{\xi \in SM \mid \pi(\xi) \in \partial M\}$

$\Rightarrow T_\xi \partial SM = \{X \in T_\xi SM \mid \pi_* X \in T_x \partial M\}$

$= \{X \in T_\xi SM \mid g(\nu_x, \pi_* X) = 0\}$

Define $N_\xi := \nu_x^k \frac{\delta}{\delta x^k}\big|_\xi$. Then $\hat g(N_\xi, N_\xi) = g(\nu_x, \nu_x) = 1$ and

$\hat g(N_\xi, X) = g(\pi_*(N_\xi), \pi_*(X)) = g(\nu_x, \pi_* X) = 0 \quad \forall X \in T_\xi \partial SM$,

so that N_ξ is the normal vector of $\partial SM \subset SM$ at ξ. $\square$

Santaló-formula

Let (M,g) be non-trapping with C^∞-boundary, and let $f \in L^1(SM)$. Then

$$\int_{SM} f(\xi)\,d\xi = \int_{\partial_- SM} \int_0^{\tau_+(\eta)} f(\Phi_H^t(\eta))\,dt\, g(\nu_-, H_\eta)\,d\eta.$$

Proof: Define $F: SM \longrightarrow \mathbb{R}\times SM$; $F(\xi) = (\tau_-(\xi), \Phi_H^{-\tau(\xi)}(\xi))$.
Then F is an injection with $\Phi_H \circ F = \mathrm{id}_{SM}$. Define

$$D = \{(t,\eta) \mid 0 < t < \tau_+(\eta),\ \eta \in \partial_- SM\}.$$

Then $F(SM)\setminus D$ and $SM \setminus \Phi_H(D)$ are of measure zero, and

$$\int_{SM} f(\xi)\,\mu_{SM} = \int_D f(\Phi_H^t(\eta))\,(\Phi_H)^*\mu_{SM}.$$

Let $X_2, \dots, X_{2n-1} \in T_\eta \partial SM$ be orthonormal. Then

$$H = \hat{g}(N_-, H_\eta)N_- + \sum_{k=2}^{2n-1} \hat{g}(X_k, H_\eta)X_k,$$

and the pull-back of the geodesic flow becomes

$$((\Phi_H)^*\mu_{SM}|_{\Phi_H^t(\eta)})\left(\frac{\partial}{\partial t}\Big|_{(t,\eta)}, X_2|_{(t,\eta)}, \dots, X_{2n-1}|_{(t,\eta)}\right)$$

$$= \mu_{SM}|_{\Phi_H^t(\eta)}\left(H|_{\Phi_H^t(\eta)}, (\Phi_H^t)_* X_2|_\eta, \dots, (\Phi_H^t)_* X_{2n-1}|_\eta\right)$$

$$= \mu_{SM}|_\eta\left(H_\eta, X_2|_\eta, \dots, X_{2n-1}|_\eta\right)$$

$$= \underbrace{\hat{g}(N_-, H_\eta)}_{=g(\nu_-,\eta)}\, \underbrace{\mu_{SM}|_\eta(N_-, X_2|_\eta, \dots, X_{2n-1}|_\eta)}_{=1}$$

$$\Rightarrow (\Phi_H^t)^*\mu_{SM}|_{\Phi_H^t(\eta)} = g(\nu_-,\eta)(dt \wedge \mu_{\partial SM})|_{(t,\eta)}. \qquad \square$$