Inverse Problems Seminar

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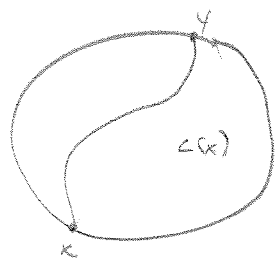
Introduction to boundary rigidity

HY + TKK joint seminar, Mon 14-16, alternates Exactum B120, U322

Lassas – Päivärinta – Salo – Somersalo

Theme: boundary rigidity, visitor lectures

Travel time tomography: measure travel times of waves, determine $c(x)$



Herglotz (1905), $n=3$

$c = c(r)$, $\frac{d}{dr}\left(\frac{r}{c(r)}\right) > 0$

(means that $c(r) < r$; speed large near origin)

Anisotropic case: $ds^2 = g_{ij}(x)\, dx^i dx^j$

$(g_{ij}(x))_{i,j=1}^n$ Riemannian metric

Boundary rigidity problem: Let (M,g) be a Riemannian mfld with boundary ∂M. If one knows the geodesic distance $d_g(x,y)$ between any two $x,y \in \partial M$, does this determine the metric g?

Here $d_g(x,y) = \inf_{\substack{\gamma:[a,b]\to M \\ \gamma(a)=x,\ \gamma(b)=y}} L(\gamma)$

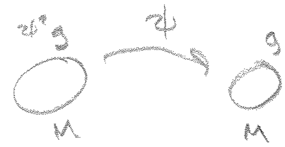
$$L(\gamma) = \int_a^b |\dot\gamma(t)|_g \, dt = \int_a^b \sqrt{\sum_{i,j} g_{ij}(\gamma(t))\, \dot\gamma^i(t) \dot\gamma^j(t)}\, dt$$

Obstructions:

Lemma If $\Psi: M \to M$ diffeo with $\Psi|_{\partial M} = id$, then $d_{\Psi^* g} = d_g$.

Pf Let $\gamma:[a,b]\to M$, $\gamma(a)=x$, $\gamma(b)=y$. Let $\tilde\gamma = \Psi^{-1}\circ\gamma$

$$L_{\Psi^* g}(\tilde\gamma) = \int_a^b \sqrt{\langle \dot{\tilde\gamma}(t), \dot{\tilde\gamma}(t)\rangle^{\Psi^* g}_{\tilde\gamma(t)}}\, dt$$
$$= \int_a^b \sqrt{\langle \Psi_* \dot{\tilde\gamma}(t), \Psi_* \dot{\tilde\gamma}(t)\rangle^{g}_{\gamma(t)}}\, dt$$
$$= \int_a^b \sqrt{\langle \dot\gamma(t), \dot\gamma(t)\rangle^{g}_{\gamma(t)}}\, dt$$
$$= L_g(\gamma)$$

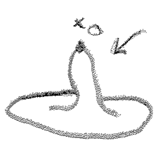


$\Psi_* v = D\Psi(x) v$

$D\Psi(x)\, D\Psi^{-1}(\Psi(x)) = I$

□

Other obstructions:



Def. Let $\mathcal{G}$ be a class of metrics on a mfld M. We say that mflds (M,g) where $g \in \mathcal{G}$ are boundary rigid if $\forall\, g_1, g_2 \in \mathcal{G}$,

$$d_{g_1}(x,y) = d_{g_2}(x,y) \quad \forall\, x,y \in \partial M \implies g_2 = \psi^* g_1 \text{ for some diffeo } \psi : M \to M,\ \psi|_{\partial M} = id.$$

Def. A mfld (M,g) is simple if
all geodesics have no conjugate points ($\forall\, x,y \in M$ $\exists!$ geodesic from x to y)
∂M is strictly convex, i.e. 2nd fund. form on ∂M is positive definite
(if $\gamma : [a,b] \to M$ geodesic joining $x,y \in \partial M$, then $\gamma((a,b)) \subseteq int(M)$)

Conjecture (Michel 1981) Compact simple mflds are boundary rigid.

Results:
- Mukhometov 1978, 1981: $d_{g_1} = d_{g_2}$, $g_2 = \alpha g_1$ ($\alpha > 0$), both simple $\implies g_1 = g_2$
- Michel 1981, $M \subseteq \mathbb{R}^2, H^2, S^2_+$ simple
- Gromov 1991, $M \subseteq \mathbb{R}^n$ simple
- Besson-Courtois-Gallot 1995, simple subspaces of symm. spaces of constant negative curvature
- Croke, Otal 1990, M 2-dim. simple negative curvature

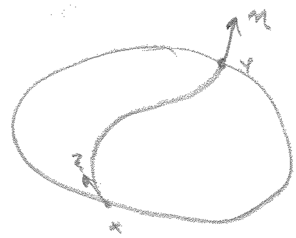
Thm (Pestov-Uhlmann 2005) Let (M, g_i), $i = 1,2$, be 2D simple compact Riemannian manifolds with boundary. Assume

$$d_{g_1}(x,y) = d_{g_2}(x,y) \quad \forall\, x,y \in \partial M.$$

Then $\exists$ diffeo $\psi : M \to M$, $\psi|_{\partial M} = id$, such that $g_2 = \psi^* g_1$.

Thm (Stefanov-Uhlmann 2005) If $n \geq 3$, uniqueness and stability for a generic set of simple metrics.

To prove 2D result need scattering relation

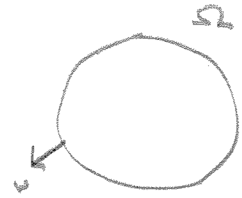


$$\alpha_g : (x,\xi) \longmapsto (y,\eta)$$
$$x,y \in \partial M, \quad |\xi|_g = |\eta|_g = 1$$
$$\alpha(x,\xi) = (\gamma(x,\xi,\tau_g^0(x,\xi)),\ \dot\gamma(x,\xi,\tau_g^0(x,\xi)))$$

For simple metrics, $d_g \implies \alpha_g$.

Conjecture If (M,g) nontrapping, does α_g (and travel time) determine g up to isometry?

We consider another problem, the inverse conductivity problem arising in EIT.



$\gamma = (\gamma^{ij})_{i,j=1}^n$ conductivity in Ω

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$$

DN map

$\Lambda_\gamma : f \mapsto -\gamma^{ij} \frac{\partial u}{\partial x_i} \nu_j |_{\partial\Omega}$, $\quad \Lambda_{\varphi^*\gamma} = \Lambda_\gamma$ if $\varphi : \Omega \to \Omega$ diffeo, $\varphi|_{\partial\Omega} = id$

Inverse problem: find (γ^{ij}) from Λ_γ (up to φ).

Geometrically, replace γ with metric g and $\nabla \cdot \gamma \nabla$ with

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left(\sqrt{\det g}\, g^{ij} \frac{\partial}{\partial x_j} u \right)$$

$$\begin{cases} \Delta_g u = 0 & \text{in } M \\ u = f & \text{on } \partial M \end{cases}$$

$$\Lambda_g : f \mapsto \langle \nu, \nabla u|_{\partial M} \rangle$$

When $n \geq 3$, $\gamma^{ij} = \sqrt{\det g}\, g^{ij}$ and the inverse problems for conductivity and metric are equivalent.

For $n = 2$, conformal invariance:

$$\Lambda_{\beta g} = \frac{1}{\beta} \Lambda_g \quad \text{where } \beta > 0$$

Therefore $\Lambda_{\beta(\varphi^* g)} = \Lambda_g$.

<u>Thm</u> (Lassas–Uhlmann 2001) If (M, g_i) connected, $\Lambda_{g_1} = \Lambda_{g_2}$ and $n = 2$, then $\exists$ diffeo $\varphi : M \to M$ with $\varphi|_{\partial M} = id$, and $\exists\ \beta \in C^\infty(M)$ with $\beta \neq 0$, $\beta|_{\partial M} = 1$, such that $g_2 = \beta \varphi^* g_1$.

<u>Thm</u> (Lassas–Taylor–Uhlmann 2003) If $n \geq 3$, $\Lambda_{g_1} = \Lambda_{g_2}$ and g_1, g_2 are real-analytic up in $\bar{M}$, then $g_2 = \varphi^* g_1$ for diffeo $\varphi : M \to M$, $\varphi|_{\partial M} = id$

Comparison of boundary rigidity and EIT

d_g / α_g	Λ_g
function of $2n-2$ variables	kernel is function of $2n-2$ variables
$d_{\varphi^* g} = d_g$	$\Lambda_{\varphi^* g} = \Lambda_g$
singularities determine $\partial^\alpha g|_{\partial M}$	singularities determine $\partial^\alpha g|_{\partial M}$
$d_g(x,y) = \inf\limits_{\substack{\gamma(a)=x \\ \gamma(b)=y}} \int_a^b \sqrt{g_{ij}(\gamma(t))\, \dot\gamma^i(t)\, \dot\gamma^j(t)}\, dt$	$Q_g(f) = \inf\limits_{u|_{\partial\Omega}=f} \int_\Omega \gamma^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}\, dx$
graph of α_g Lagrangian mfld (finite dim.)	graph of Λ_g Lagrangian mfld (infinite dim.)
obtained from solutions of Hamilton ODE	obtained from solution of PDEs

Surprising result is

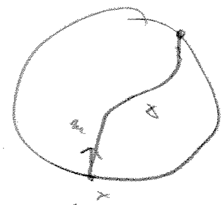
Thm (Pestov–Uhlmann 2005) (M, g_i) compact simple 2D mflds.
Then $\alpha_{g_1} = \alpha_{g_2} \Rightarrow \Lambda_{g_1} = \Lambda_{g_2}$.

Given this, proof that $d_{g_1} = d_{g_2} \Rightarrow g_2 = \psi^* g_1$ for simple 2D mflds goes as follows:

$$d_{g_1} = d_{g_2}$$
$$\overset{\text{simple}}{\Longrightarrow} \quad \alpha_{g_1} = \alpha_{g_2}$$
$$\overset{\text{Pestov-Uhlmann}}{\Longrightarrow} \quad \Lambda_{g_1} = \Lambda_{g_2}$$
$$\overset{\text{Lassas-Uhlmann}}{\Longrightarrow} \quad g_2 = \beta(\psi^* g_1)$$
$$\overset{\text{Mukhometov}}{\Longrightarrow} \quad g_2 = \psi^* g_1 .$$

Geodesic X-ray transform

$$I_g f(\gamma) = \int_\gamma f = \int_0^{\tau(x,\xi)} f(\gamma(t,x,\xi))\,dt$$



Linearization of boundary rigidity problem about a simple metric results in injectivity questions for $I_g : C^\infty(M) \to C^\infty(\partial_+\Omega(M))$.
Need that I^* onto.

Open problems:

1) Nonsmooth metrics ($g \in C^{1,1}$)?
2) Reconstruction of g from d_g
3) $n=2$, g nontrapping, does α_g determine g?
4) g nontrapping, is I_g injective?
5) Is there a connection between Λ_g and d_g?
6) $n \geq 3$, connection between d_g and Λ_g? (Gromov's result for Λ_g)