Riemannian geometry
Exercises 2, 10.11.2015

1. Compute the expression of the Riemannian metric of $\mathbb{E}^2$ in polar coordinates.

2. Show that the mappings $x \mapsto (-x_1, x_2)$ and $x \mapsto \frac{x}{\|x\|^2}$ are Riemannian isometries of the hyperbolic plane.

3. A Riemannian manifold (M, g) is called *a symmetric spaces* if for all $p \in M$ there is a Riemannian isometry of (M, g) ι_p such that $d\iota_p|_{T_pM} = -\mathrm{id}$. Show that the Riemannian manifolds $\mathbb{E}^2$, $\mathbb{S}^2$ and $\mathbb{H}^2$ are symmetric spaces.

4. Let (M, g) be a Riemannian manifold and let $p \in M$. Let (U, x) and (V, y) be local coordinates such that $p \in U \cap V$. What is the relation between the coordinate expressions of g in these two coordinates?

5. Show that the directional derivative $(X, Y) \mapsto DY\,X$ is the Levi-Civita connection of the Riemannian manifold $\mathbb{E}^n$. It is enough to show that it has no torsion and that it is compatible with the Euclidean Riemannian metric.

6. Let $X, Y \in \mathscr{F}(M)$. Show that the mapping $L_{X,Y} \colon \mathscr{X}(M) \to \mathscr{F}(M)$,

$$L_{X,Y}(Z) = X\,g(Y,Z) + Y\,g(Z,X) - Z\,g(X,Y) - g(Y,[X,Z]) - g(Z,[Y,X]) + g(X,[Z,Y])$$

is $\mathscr{F}(M)$-linear.