Riemannian geometry
Exercises 1, 3.11.2015

1. Show that every 1-form θ can be written in local coordinates using coordinate 1-forms as

$$\theta = \sum_i \theta\partial_i \, dx^i.$$

2. Prove the claim of Proposition 1.5 for $(1,2)$-tensors.

3. The exterior derivative $d\theta$ of a 1-form $\theta \in \mathscr{X}^*(M)$ is defined by

$$d\theta(X,Y) = X\theta Y - Y\theta X - \theta[X,Y]$$

for all $X, Y \in \mathscr{X}(M)$. Show that $d\theta$ is a tensor.

4. Let M_1be a differentiable manifold, let (M_2, g_2) be a Riemannian manifold jand let $\phi\colon M_1 \to M_2$ be an immersion. Show that $\phi^* g_2$ is a Riemannian metric.

5. Let (M, g) be a Riemannian manifold, let $\gamma\colon [a,b] \to X$ be a smooth path and let $\phi : [a', b'] \to [a, b]$ be a diffeomorphism. Show that the length of $\gamma \circ \phi$ equals the length of γ.

6. Let $z_k = (x_k, y_k) \in \mathbb{H}^2$ be a sequence such that y_k is not bounded. Show that z_k is not a Cauchy sequence.

7. Let (M, g) be a Riemannian manifold. Let $\phi\colon \mathscr{X}(M) \to \mathscr{X}^*(M)$ be the mapping defined by $\Phi(V) = V^*$, where

$$V^*(X) = g(V, X)$$

for all $X \in \mathscr{X}(M)$. Show that Φ is a $\mathscr{F}(M)$-linear isomorphism.

[7]Vihje: When proving that the map is onto, present a 1-form θ using coordinate 1-forms in a coordinate neighbourhood U and find an expression for a vector field Y such that $\Phi(Y|_U) = \theta|_U$.