Number theory 2 2024

Exercises 5

Rational numbers $\frac{a}{c}, \frac{b}{d} \in \mathbb{Q}$ are *adjacent*, if $|ad - bc| = 1$.

1. Determine the rational numbers that are adjacent to $\frac{2}{3}$.

2. Determine the rational numbers that are adjacent to $\frac{23}{41}$.

3. Let $\frac{a}{c}$ and $\frac{b}{d}$ be adjacent. Prove that $\frac{b+ka}{d+kc}$ is adjacent to $\frac{b+(k+1)a}{d+(k+1)c}$.

If $\frac{a}{c}$ and $\frac{b}{d}$ are adjacent, the rational number $\frac{a+b}{c+d}$ is their *mediant*.

4. Let $r, s \in \mathbb{Q} \cup \{\frac{1}{0}\}$ be adjacent. Prove that there are exactly two elements of $\mathbb{Q} \cup \{\frac{1}{0}\}$ and that one of these is the mediant of r and s.

Let $\frac{p}{q} \in \mathbb{Q}$ with $\gcd(p, q) = 1$. The *Ford disk* at $\frac{p}{q}$ is

$$\mathscr{H}_{\frac{p}{q}} = \left\{ (x, y) \in \mathbb{R}^2 : \left\| w - \left(\frac{p}{q}, \frac{1}{2q^2} \right) \right\| \leq \frac{1}{2q^2} \right\}.$$

The boundary $\partial \mathscr{H}_{\frac{p}{q}}$ of $\mathscr{H}_{\frac{p}{q}}$ is the *Ford circle* at $\frac{p}{q}$.
The half-plane

$$\mathscr{H}_{\frac{1}{0}} = \{(x, y) \in \mathbb{R}^2 : y \geq 1\}$$

is the *Ford disk at infinity*.

5. Let $\frac{a}{c}, \frac{b}{d} \in \mathbb{Q} \cup \{\infty\}$. Prove that the Ford disks $\mathscr{H}_{\frac{a}{c}}$ and $\mathscr{H}_{\frac{b}{d}}$ are tangent if $\frac{a}{c}$ is adjacent to $\frac{b}{d}$, and that the disks are disjoint if $\frac{a}{c}$ is not adjacent to $\frac{b}{d}$.

If $r, s, t \in \mathbb{Q} \cup \{\infty\}$ are pairwise adjacent, then the compact region in the complement of $\mathscr{H}_r$, $\mathscr{H}_s$ and $\mathscr{H}_t$ in the plane $\mathbb{R}^2$ is the *Ford triangle* $\Delta(r, s, t)$.

6. Let

$$\begin{aligned} A = (A_1, A_2) &= \mathscr{H}_{\frac{p}{q}} \cap \mathscr{H}_{\frac{r}{s}}\,, \\ B = (B_1, B_2) &= \mathscr{H}_{\frac{r}{s}} \cap \mathscr{H}_{\frac{t}{u}} \quad \text{and} \\ C = (C_1, C_2) &= \mathscr{H}_{\frac{t}{u}} \cap \mathscr{H}_{\frac{p}{q}} \end{aligned}$$

be the vertices of the Ford triangle $\Delta = \Delta(\frac{p}{q}, \frac{r}{s}, \frac{t}{u})$ as in Figure 0.1. Prove that

$$A_1 = \frac{pq + rs}{q^2 + s^2}\,, \quad B_1 = \frac{rs + tu}{s^2 + u^2} \quad \text{and} \quad C_1 = \frac{tu + pq}{u^2 + q^2}\,,$$

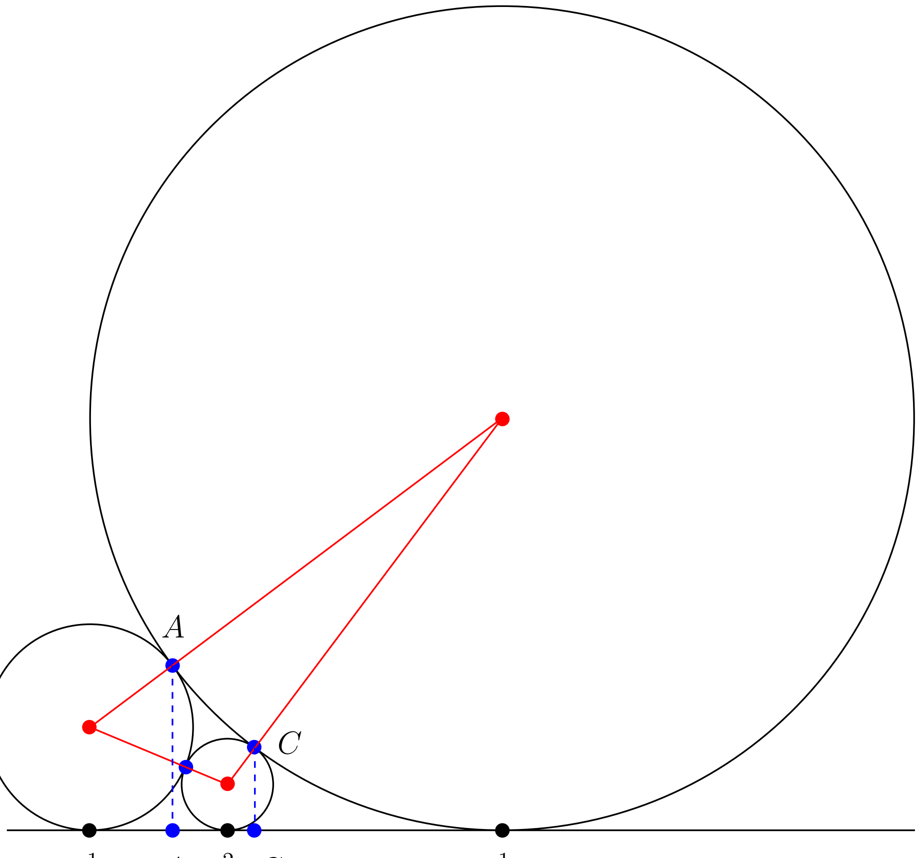

Figure 0.1: $\frac{p}{q} = \frac{1}{1}$, $\frac{r}{s} = \frac{1}{2}$ and $\frac{t}{u} = \frac{2}{3}$.

7. Find four rational solutions to the inequality

$$\left|\frac{1}{\sqrt{2}} - \frac{p}{q}\right| < \frac{1}{2q^2}$$

with the help of Figures 0.2 0.3.

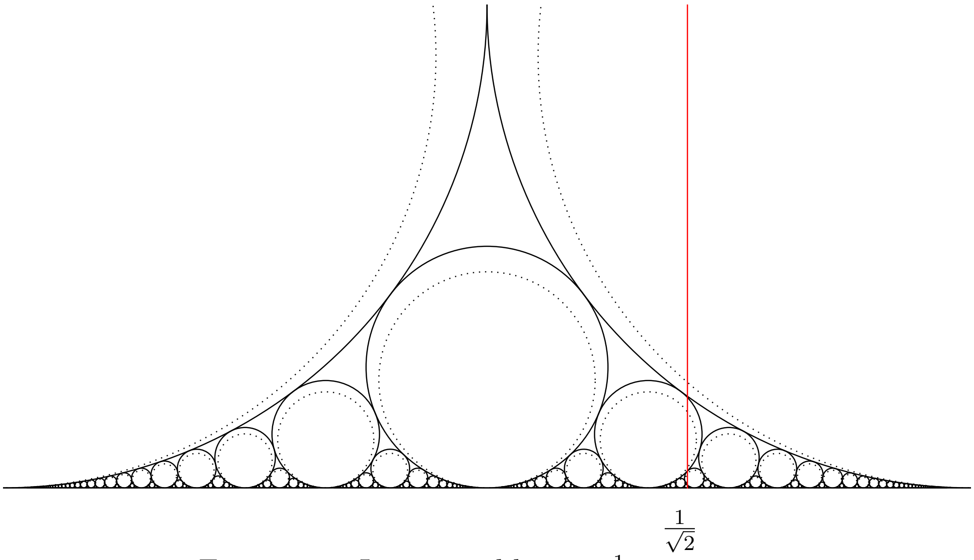

Figure 0.2: Irrationaaliluvun $\frac{1}{\sqrt{2}}$ arviointia.

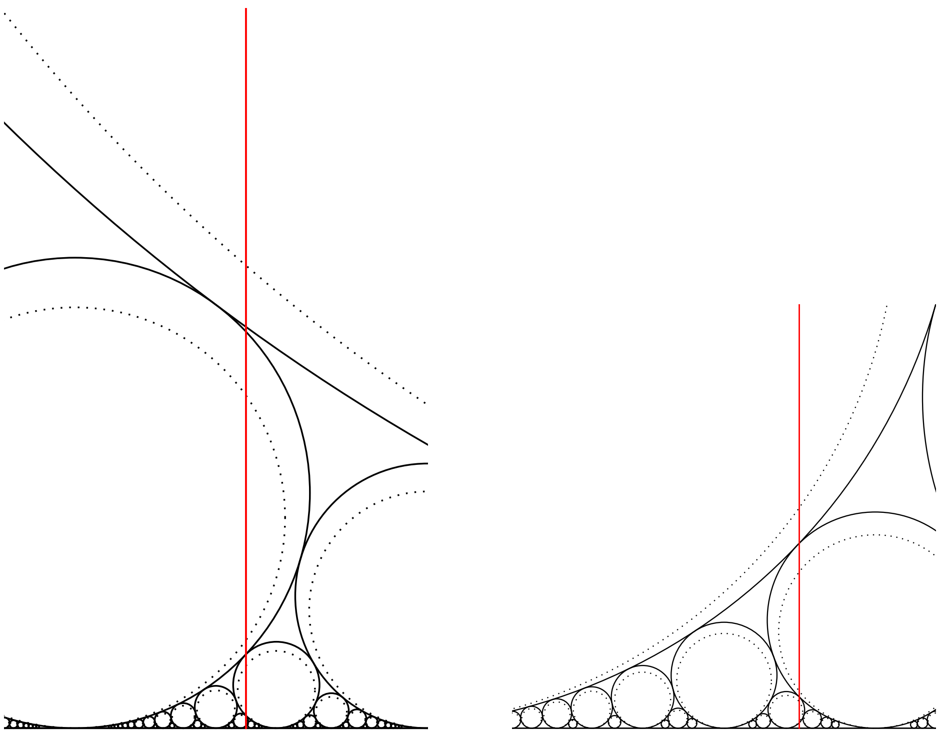

Figure 0.3: Irrationaaliluvun $\frac{1}{\sqrt{2}}$ arviointia, lähikuvia.