Number theory 2 2024

Exercises 1

1. Determine the number of unrestricted partitions of 6.[1]

Solution. There are 11 partitions:

$$\begin{aligned}6 &= 5+1 = 4+2 = 4+1+1 = 3+3 = 3+2+1 = 3+1+1+1\\ &= 2+2+2 = 2+2+1+1 = 2+1+1+1+1 = 1+1+1+1+1+1\end{aligned}$$

2. Prove that a positive natural number n can be represented as the sum of positive integers in 2^{n-1} when two sums with the same numbers in a different order are counted separately.[2]

Solution. Place n stones in a row. There are $n-1$ gaps between the stones where one can choose to place a separator or no separator. Each way to place from 0 to $n-1$ separators is identified with a binary number $(a_{n-2}a_{n-3}\cdots a_i a_0)_2$, and all binary numbers from 0 to $(11\cdots 11)_2 = \sum_{i=0}^{n-2} i^2 = 2^{n-1}-1$ can be realized in this way.

3. Let $a, b, c, d \in \mathbb{Z}$. Prove that

$$(a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad+bc)^2$$

Solution. Expand both expressions an compare.

4. Represent 10285 as a sum of two squares.

Solution. The prime decomposition is $10285 = 5 \cdot 11^1 \cdot 17$. Exercise 3 gives

$$5 \cdot 17 = (2^1 + 1^2)(4^2 + 1^2) = 2^2 + 9^2\,,$$

and, therefore,

$$10285 = 11^2(2^2 + 9^2) = 22^2 + 99^2\,.$$

If you do the computations in a different order, you may also get

$$10285 = 66^2 + 77^2\,.$$

5. Represent 1517 as a sum of two squares.

Solution. The prime decomposition is $1517 = 37 \cdot 41$. Exercise 3 gives

$$1517 = 37 \cdot 41 = (6^2 + 1^2)(5^2 + 4^2) = 26^2 + 29^2\,.$$

If you do the computations in a different order, you may also get

$$1517 = 19^2 + 34^2\,.$$

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[1] See for example §19.2 of Hardy and Wright.

[2] Place n stones in a row. There are $n-1$ gaps between the stones where one can choose to place a separator or no separator. This choice can be coded by the numbers 1 and 0...

6. Which of the numbers 105, 2205 and 5951 are sums of two squares?[3]

Solution. The primes 3 and $7 \equiv 3 \bmod 4$ appear with odd exponent 1 in the prime decomposition $105 = 3 \cdot 5 \cdot 7$. By the sum of two squares theorem, 105 is not the sum of two squares.

The primes 3 and $7 \equiv 3 \bmod 4$ appear with even exponent 2 in the prime decomposition $2205 = 3^2 \cdot 5 \cdot 7^2$. By the sum of two squares theorem, 2205 $(= 21^2(2^2+1^2) = 42^2+21^2)$ is the sum of two squares.

The alternating sum of digits test for divisibility shows that $5951 = 11 \cdot 541$ and that 541 is not divisible by $11 \equiv 3 \mod 4$. By the sum of two squares theorem, 5951 is not the sum of two squares. It is not necessary to check that 541 is, in fact, a prime.

[3]It is not required to give the representations.