Neg. curved spaces 5.11.2020

$\rho_1, \rho_2 : [0,\infty[\longrightarrow X$ asymptotic if

$$\sup_{0 \le t < \infty} d(\rho_1(t), \rho_2(t)) < \infty$$

$$\Longleftrightarrow \quad d_{Haus}\big(\rho_1([0,\infty[), \rho_2([0,\infty[)\big) < \infty .$$ (Prop 8.3)

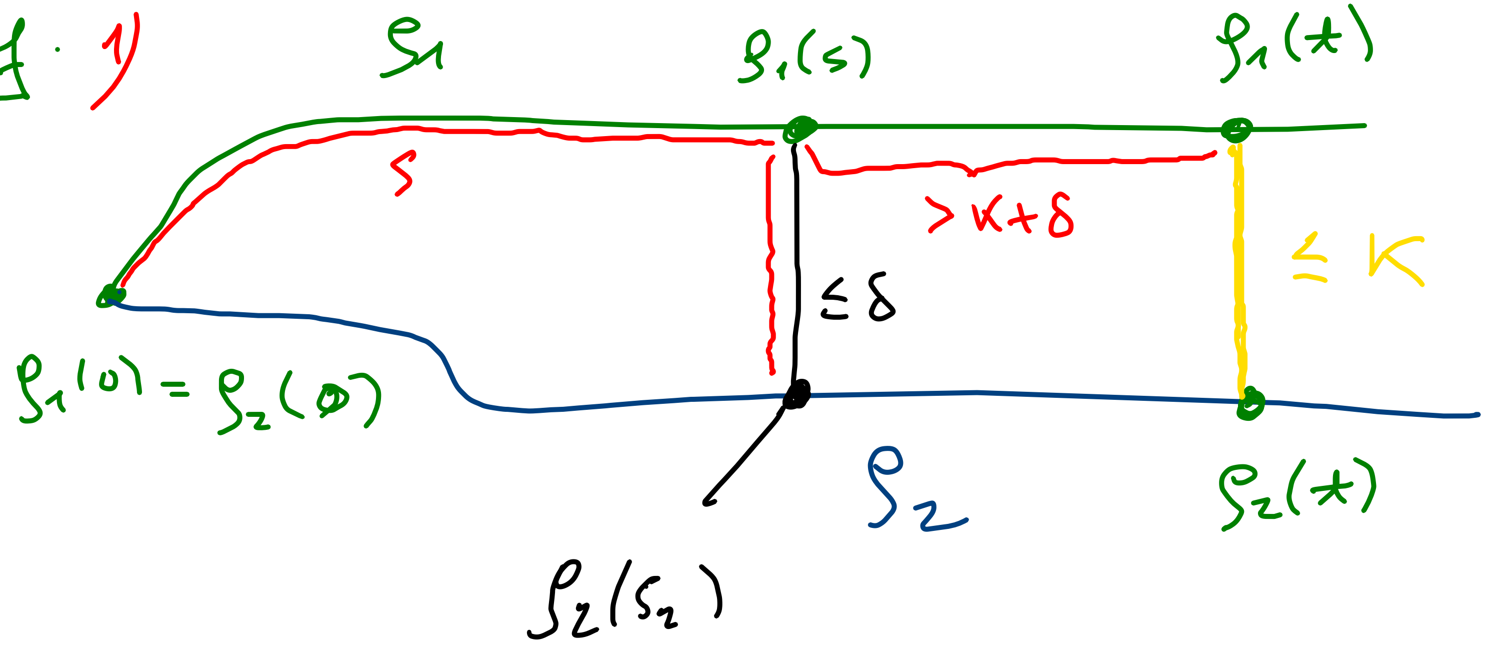
Prop. 8.4 X δ-hyp, $\rho_1, \rho_2 \in \mathcal{G}_+(X)$ asymptotic.

1) If $\rho_1(0) = \rho_2(0)$, $d(\rho_1(t), \rho_2(t)) \le 2\delta$.

2) For large t $\exists\, s_t$ s.t. $d(\rho_1(t), \rho_2(s_t)) \le 2\delta$

3) ——||—— $\exists\, u \in \mathbb{R}$ s.t. $d(\rho_1(t), \rho_2(t+u)) \le 6\delta$.

Proof. 1)

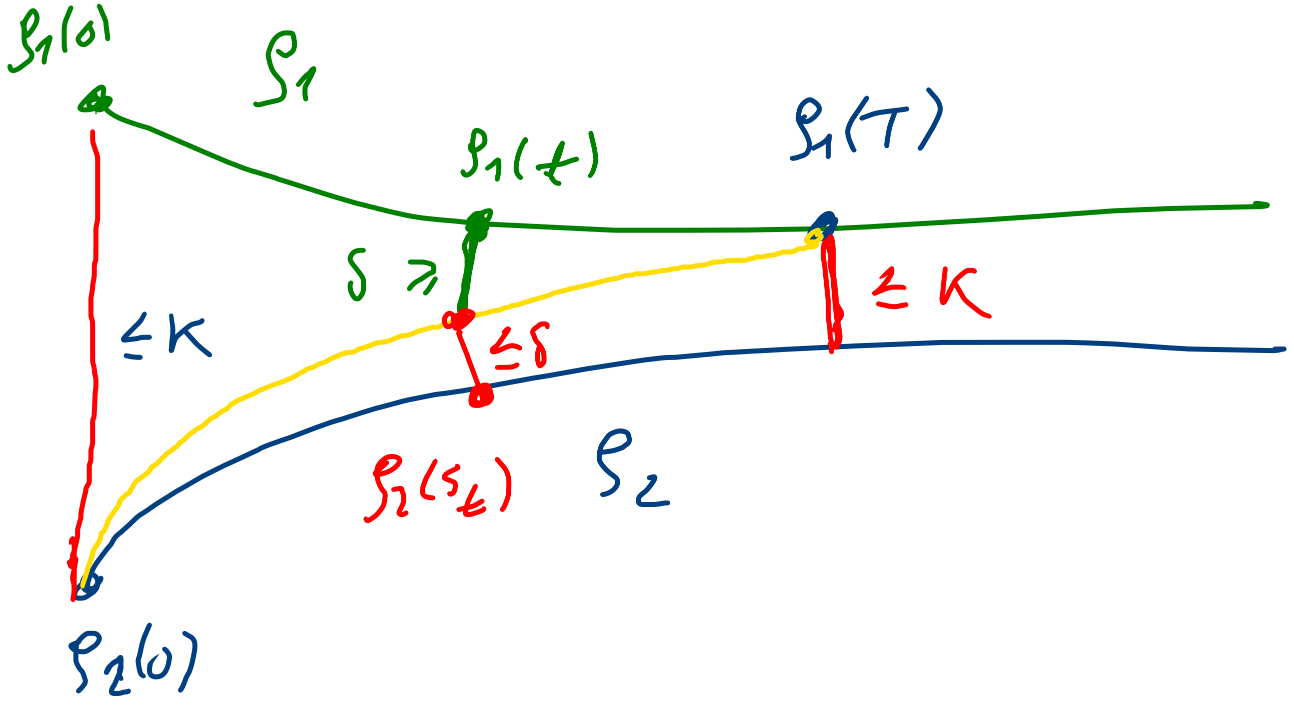


$$\boxed{K = \sup_{0 \le t < \infty} d(\rho_1(t), \rho_2(t)) < \infty}$$

Δ-ineq $\Rightarrow$ $|s_2 - s| \le \delta$

$d(\rho_1(s), \rho_2(s)) \le \delta + \delta = 2\delta.$

2)



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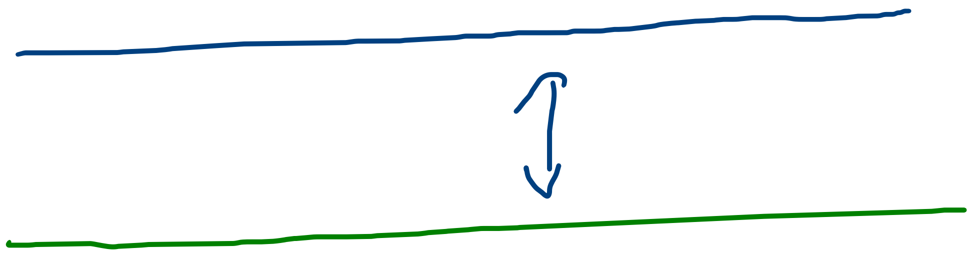
3) See notes.

Note E^2



distance of parallel lines can be big.

Defn The $\left\{\begin{array}{l}\text{boundary at infinity} \\ \text{space at infinity} \\ \text{Gromov boundary}\end{array}\right\}$ of X is $\partial_\infty X = \mathcal{G}_+(X)/\sim$

asymptoticity

Note: Prop. 8.2: Same defn as in Sect. 5.3 for the Poincaré and UHS models of $\mathbb{H}^n$.

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asymptoticity class of ρ

Lemma 8.5. $g \in \operatorname{Isom} X$, $\rho \in \mathcal{G}_+(X)$ $\boxed{g \cdot \tilde{\rho}(\infty) = (g \circ \rho)(\infty)}$

defines an action of $\operatorname{Isom} X$ on $\partial_\infty X$. $\quad \partial_\infty X \to \partial_\infty X$

Proof. Isometries preserve asymptoticity. OK. $\rightsquigarrow$ $\rho(\infty) \mapsto g \cdot \rho(\infty)$ is a bijection.

$$(g_1 \cdot g_2) \cdot \rho = g_1 \circ (g_2 \circ \rho) \Longrightarrow \text{we have an action.} \quad \square$$

$\mathcal{G}(X) = \{ g : \mathbb{R} \to X \text{ geod. line} \}$.

$g \in \mathcal{G}(X) \rightsquigarrow \rho_g^+ = g|_{[0,\infty[}$ ⎫ geod. rays

positive ray of g

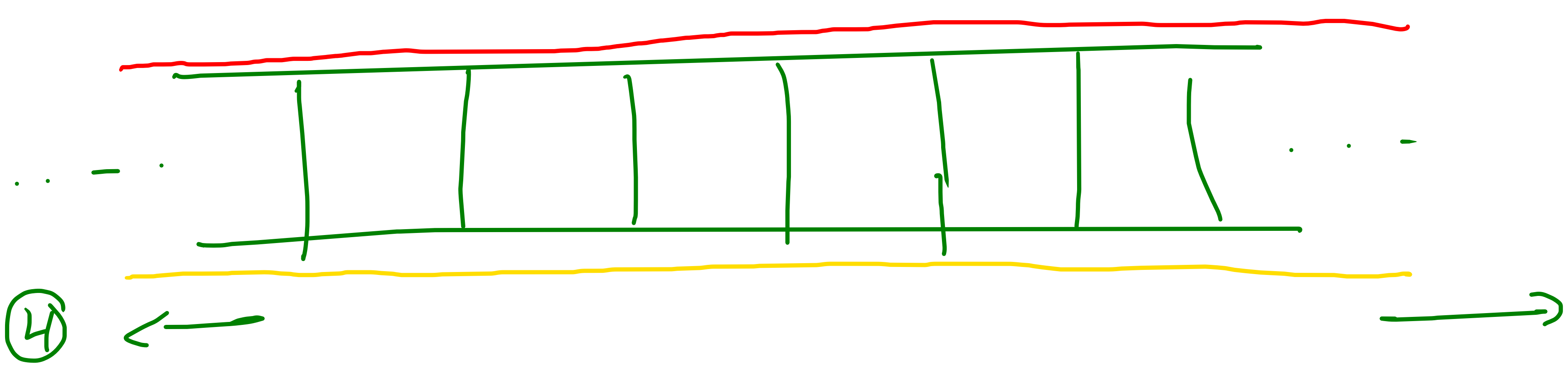
negative ray

$\rho_g^- : t \mapsto g(-t)$ ⎭

$\rightsquigarrow$ endpoints of g are the $\left\{\begin{array}{l}\text{asympt. classes} \\ \text{points at infinity}\end{array}\right\}$ of ρ_g^+ and ρ_g^-

$\rho_g^+(\infty)$ and $\rho_g^-(\infty)$.

bi-infinite ladder



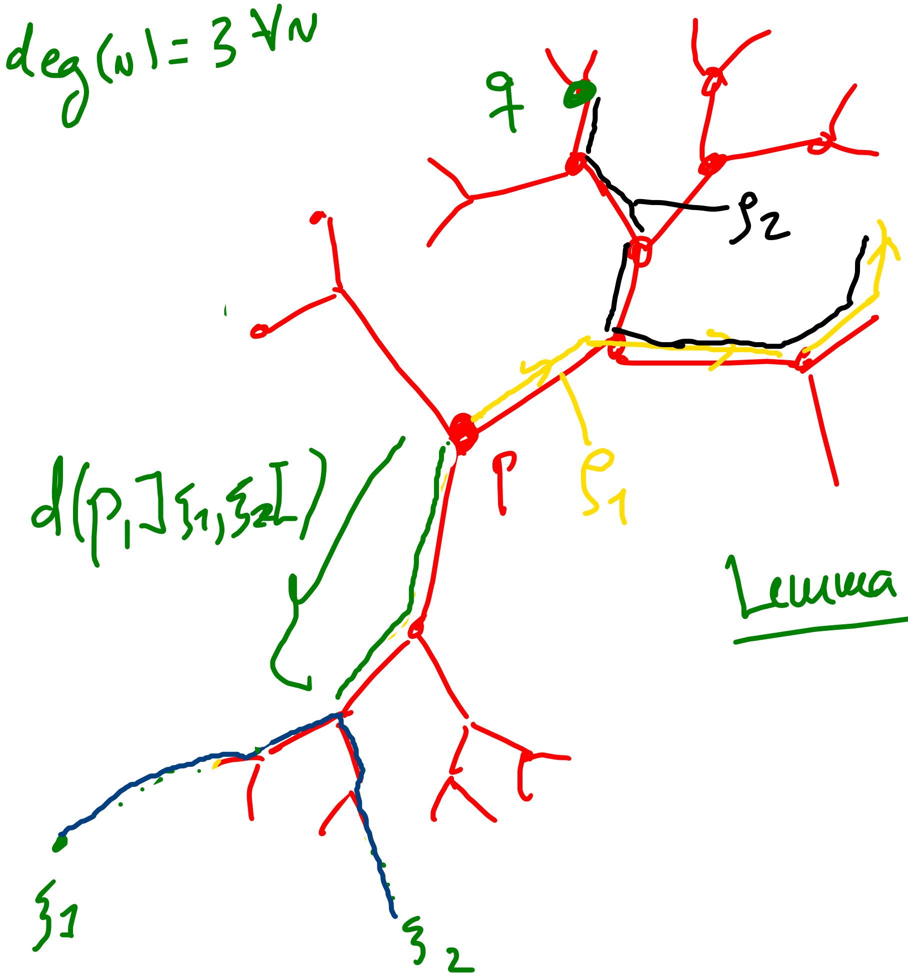
If for $\xi_1, \xi_2 \in \partial_\infty X$

$\exists_1$ geod. line (up to translation of the domain of definition) with endpoints ξ_1, ξ_2 we denote this line by $]\xi_1, \xi_2[$.

the image of

The boundary at infinity of a tree

$\deg(v) = 3\ \forall v$



Simplicial tree X, $\boxed{\deg(v) \geq 3}\ \forall\, v \in VX$

Lemma 8.8. Two rays in T are asymptotic iff $\exists\, T_1, T_2$ s.t. $\rho_1(t) = \rho_2(t + T_2)\ \forall\, t > T_1$.

Lemma 8.9 If $\rho \in \mathcal{G}_+X$ and $q \in VX$, then $\exists_1\, \rho_q \in \mathcal{G}_+X$ s.t. $\rho_q(0) = q$ and $\rho_q(\infty) = \rho(\infty)$.

(In the picture $\rho = \rho_1$, $\rho_q = \rho_2$)

$\longrightarrow\ \partial_\infty X = \mathcal{G}_+(X, p)$ for $p \in VX$.

Lemma 8.10. $\xi_1, \xi_2 \in \partial_\infty X$, $\xi_1 \neq \xi_2$, $\Rightarrow \exists_1$ up to transl. of domain of defn. geod. line with endpts. ξ_1 neg. endpt, ξ_2 pos. endpt.

$$d(p,]\xi_1(\infty), \xi_2(\infty)[) = \underline{\underline{\lim_{t\to\infty} (\xi_1(t) | \xi_2(t))_p}} = (\xi_1(\infty)|\xi_2(\infty))_p.$$

Gromov product in $\partial_\infty X$.

Def$^{\underline{n}}$ $\quad d_p(\xi_1, \xi_2) = e^{-(\xi_1|\xi_2)_p}$

Lemma 8.11. d_p is $\begin{cases} \text{a metric} \\ \text{an } \underline{\text{ultrametric}} \end{cases}$ (triangle inequality is replaced by the stronger ultrametric ineq:

$$d(x,y) \le \max(d(x,z), d(z,y)) \quad \forall z$$)

Proof. Exercise.

Prop. 8.12. X simplicial tree, $\boxed{\deg(v) \geq 3}$ $\forall v \in VX$, $p \in VX$.

1) $(\partial_\infty X, d_p)$ is totally disconnected and perfect.
no nontrivial conn. subsets — no isolated points.

2) If $\deg(v) < \infty$ $\forall v \in V$, then $(\partial_\infty X, d_p)$ is compact.

Proof. HY Cor. 2-58.

Cor. If $\deg(v) \geq 3$ and bounded then $(\partial_\infty X, d_p)$ homeo. Cantor set.

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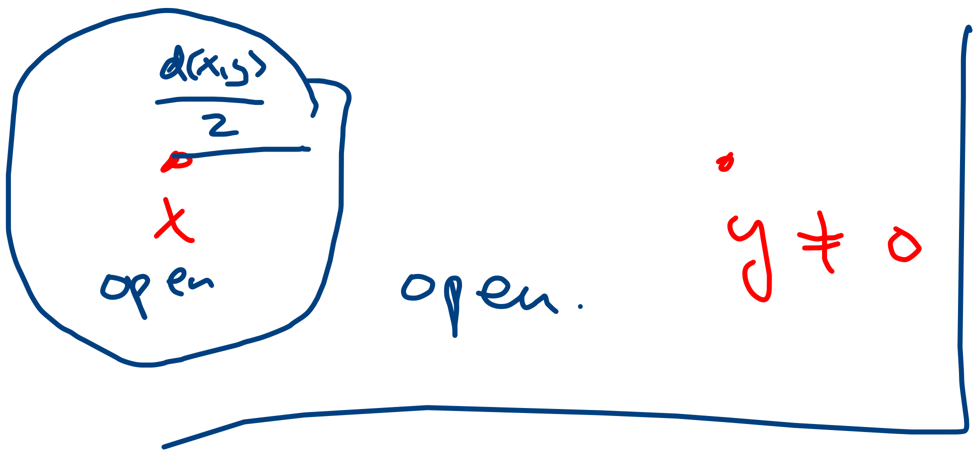
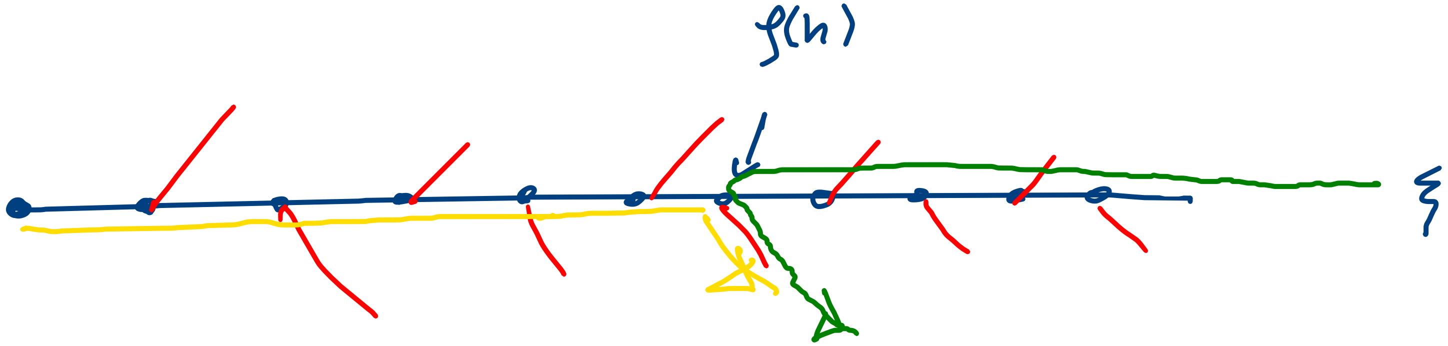
Proof. 1) All ultrametric spaces are totally disconnected b/c open balls are closed.





Let $\xi = \rho(\infty) \in \partial_\infty X$. Let $\rho_n \in \mathcal{G}_+(X,p)$ s.t.

$$\rho_n([0,n]) = \rho([0,n])$$

$$\rho_n(]n,\infty[) \cap \rho(]n,\infty[) = \emptyset$$

$$d(\xi, \rho_n(\infty)) = e^{-d(p,]\xi, \rho_n(\infty)[)} = e^{-n} \xrightarrow[n\to\infty]{} 0.$$

$$\Rightarrow \rho_n(\infty) \xrightarrow{n\to\infty} \xi. \quad \square$$

2/ Ex.