Neg. curved spaces 4.11.2020

Thm 7.23 (Švarc, Milnor) X proper geodesic space, $G \curvearrowright X$ by isom, cocompactly and properly. Then G is finitely generated and $g \mapsto g \cdot x_0$ is a quasi-isometry for any $x_0 \in X$.

$g \in G$

$d_S(g_1, g_2)$ word metric
$= \min \{ n \in \mathbb{N} : g_1^{-1} g_2 = s_1 \cdots s_n, \; s_i \in S \}$,
S symm. gen. set of G

Def^n $G \curvearrowright X$ properly if $\forall K \subset X$ compact
$\# \{ g \in G : g \cdot K \cap K \neq \emptyset \} < \infty$

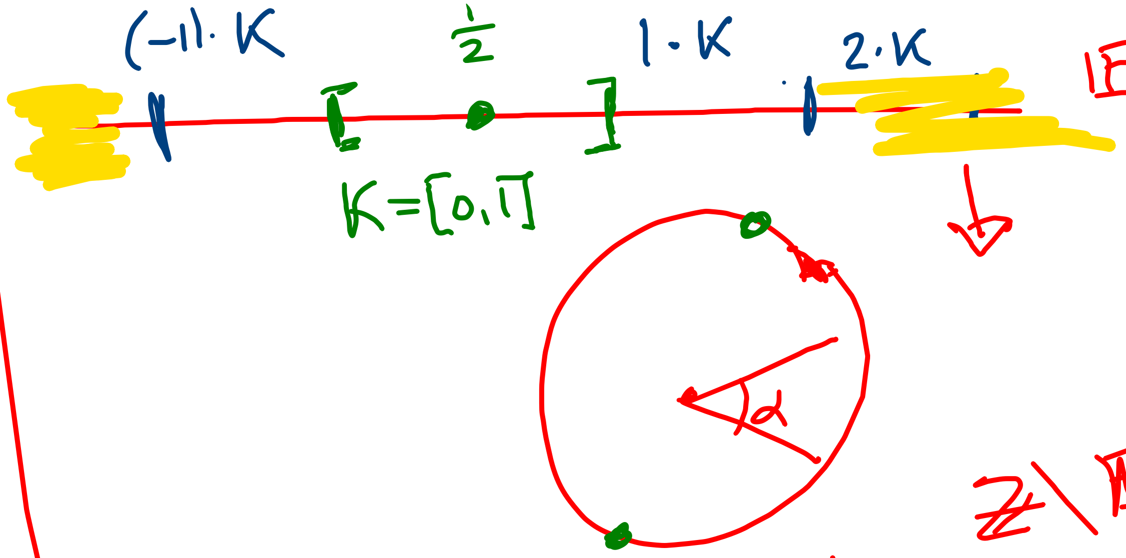
Proof of Thm 7.23: Recall Lemma 7.20: $d(g_1 \cdot x_0, g_2 \cdot x_0) \leq M \, d_S(g_1, g_2)$

G acts cocompactly $\Rightarrow (G \backslash X, \bar{d})$ compact $\Rightarrow$ finite diameter $R = \operatorname{diam}(G \backslash X)$

quotient metric

Let $x_0 \in X$, $K = \overline{B}(x_0, R)$

$$X = \bigcup_{g \in K} g \cdot K .$$



$\mathbb{E}^1 \curvearrowleft \mathbb{Z}$

$\mathbb{Z} \backslash \mathbb{E}^1 = S^1$

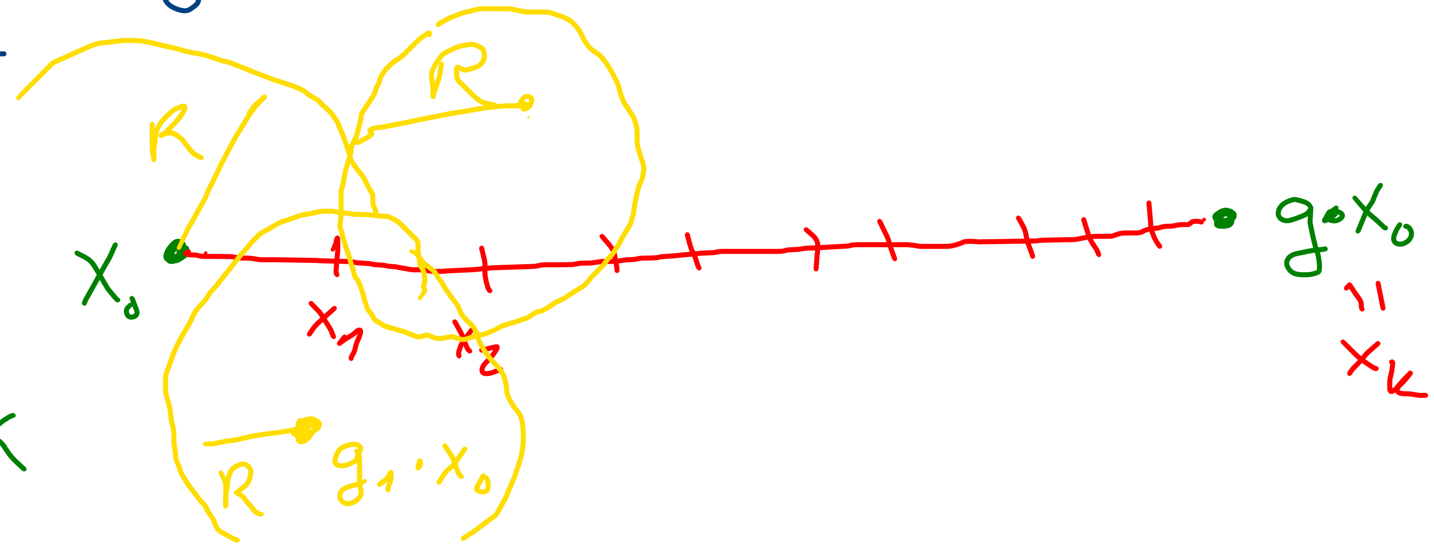
$\mathrm{diam}(\mathbb{Z}\backslash\mathbb{E}^1) = \frac{1}{2}$

$S = \{ g \in G : g\cdot K \cap K \neq \emptyset \} - \{e\}$.

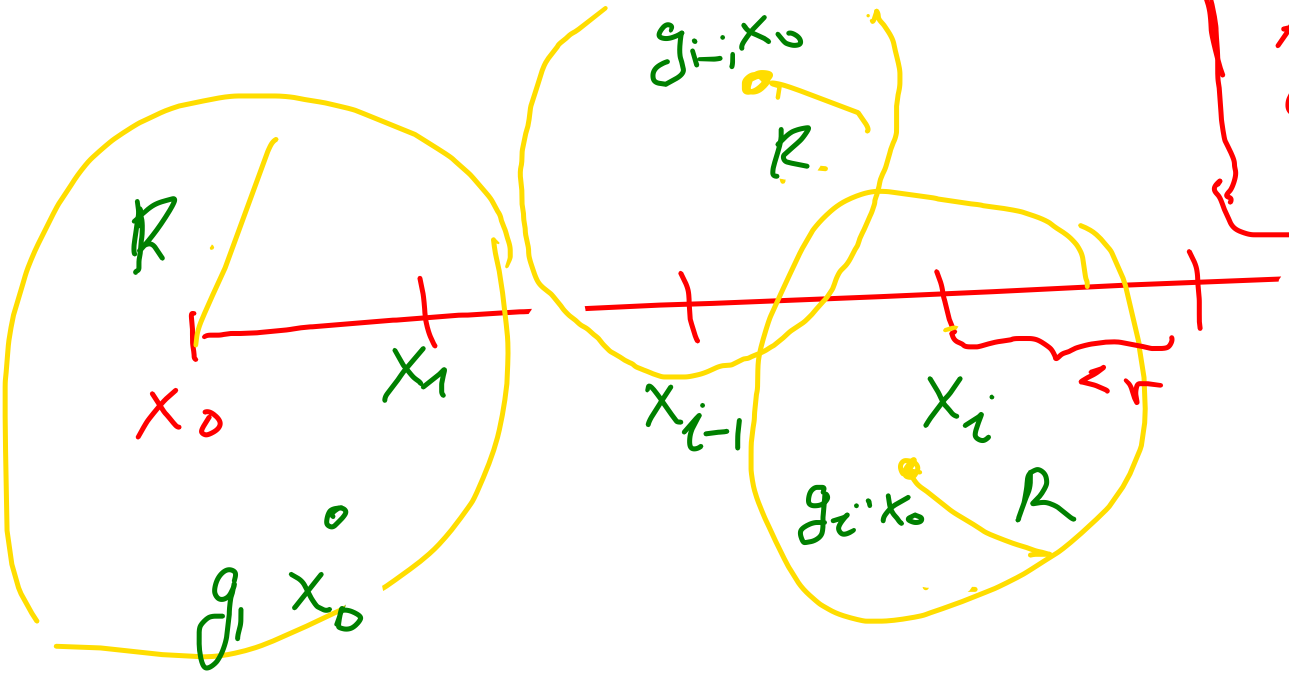
S is symmetric: $x \in g\cdot K \cap K \Rightarrow g^{-1}\cdot x \in K \cap g^{-1}\cdot K$.

$$r = \min\{ \underbrace{d(K, g\cdot K)}_{>0} : g \in G - (S \cup \{e\}) \} > 0$$

Let $g \in G - (S \cup \{e\})$



② $\exists\, g_1, \dots, g_k$: $x_i \in g_i \cdot K$
$g_0 = e$.



Assume further that $d(x_{i-1}, x_i) < R$
Choose k minimal with this property

$\leadsto k \le \frac{d(x_1, g\cdot x_0)}{R} + 1$

$x_k = \underline{g\cdot x_0} = g_k \cdot x_0$

$\Rightarrow d_S(e,g) \le \frac{d(x_0, g\cdot x_0)}{R} + 1$

$s_i = g_{i-1}^{-1} g_i$. $\quad g_{i-1}^{-1}(g_{i-1}\cdot x_0) = x_0 \Rightarrow \underline{g_{i-1}^{-1}(x_{i-1})} \in K$.

$$\boxed{d(K, s_i\cdot K) \le d(g_{i-1}^{-1}\cdot x_{i-1}, \underbrace{s_i g_i^{-1}}_{g_{i-1}^{-1}} x_i) = d(x_{i-1}, x_i) < r}$$

$\Rightarrow s_i \in S\cup\{e\}$.

$\underline{g = g_k} = \cdots g_{k-2}^{-1} g_{k-1} g_{k-1}^{-1} g_k = \underline{s_1 s_2 \cdots s_k} \leadsto S$ is a generating set.

$\Rightarrow$ $g \mapsto g \cdot x_0$ is a q.i. embedding.

Prop 7.6 $\Rightarrow$ $g \overset{\Phi}{\mapsto} g \cdot x_0$ is a quasi-isometry

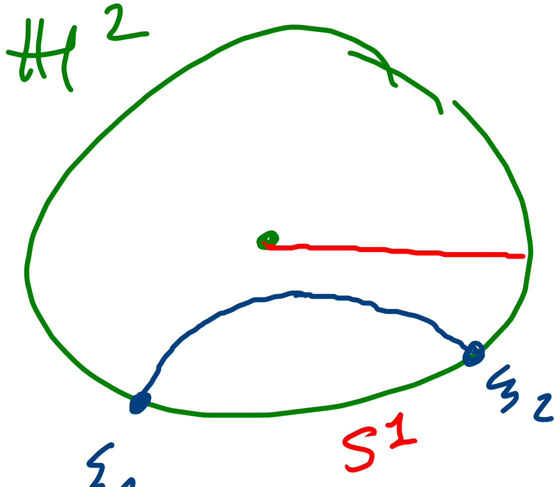
b/c $X = \bigcup_{g \in G} g \cdot K \rightsquigarrow d(x, \Phi(G)) \leq R$

$K = \overline{B}(x_0, R)$ $\square$

8 The boundary at infinity

Recall $\mathbb{H}^2$

Poincaré model $= (B(0,1), d_p)$



$\forall\ \xi_1, \xi_2 \in S^1$ $\exists_1$ geod. line with endpoints ξ_1, ξ_2

intersection pts of the associated Eucl. objects with S^1.

we know geod lines in $\mathbb{H}^2$ are arcs of circles or Eucl. segments orthog to S^1

Now: a proper intrinsic definition of endpts of geod rays and lines.

(4)

$\mathcal{G}_+(X) = \{ \text{geod. rays } \rho : [0,\infty[\to X \}$.

$\mathcal{G}_+(X,p) = \{ \rho \in \mathcal{G}(X) : \rho(0) = p \}$.

equivalence rel$\underline{\text{n}}$ OK.

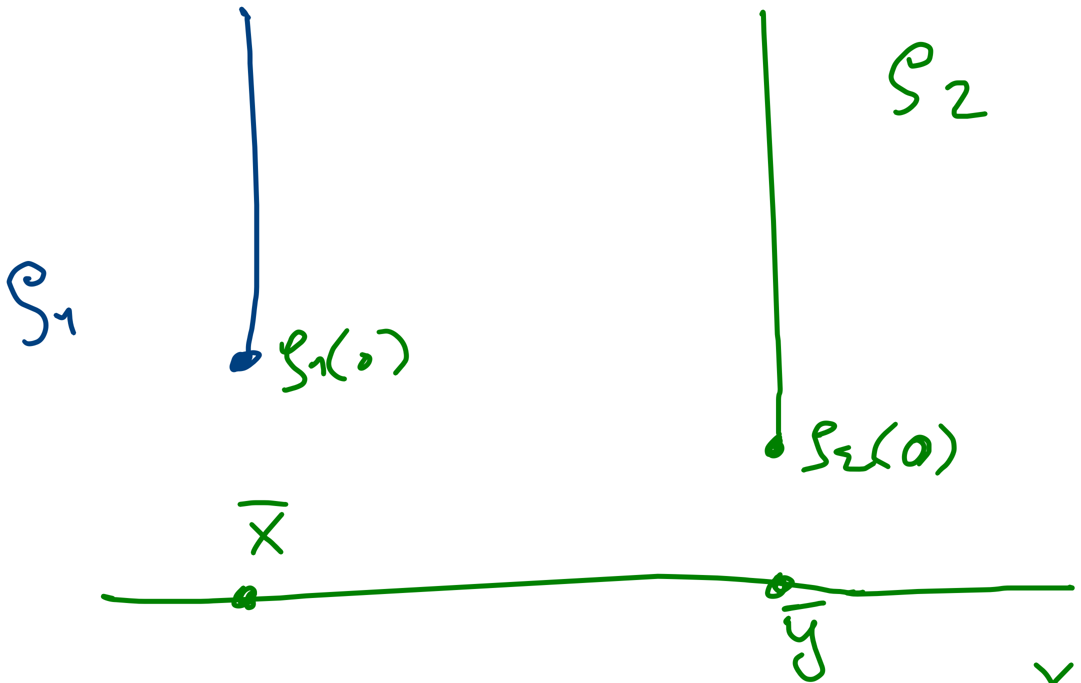
$\rho_1, \rho_2 \in \mathcal{G}_+(X)$ are <u>asymptotic</u>, $\rho_1 \sim \rho_2$, if

$$\sup_{0 \le t < \infty} d(\rho_1(t), \rho_2(t)) < \infty.$$

$\boxed{[\rho] = \rho(\infty)}$ is the equivalence class of a ray $\rho \in \mathcal{G}_+(X)$.

Prop. 8.2 $\rho_1, \rho_2 \in \mathcal{G}_+(\mathbb{H}^n)$. $\rho_1 \sim \rho_2 \iff \rho_1$ and ρ_2 have the same endpoint in the Poincaré model or the UHS model.

Proof Ch. 5: $\exists$ natural bijection from Poincaré model to UHS model that preserves having same endpoint $\rightsquigarrow$ can restrict to use UHS model.



$\rho_1, \rho_2 : [0,\infty[\to \mathbb{H}^n$.

Assume that ρ_1, ρ_2 have the same endpt in UHS. Assume endpt $= \infty$.

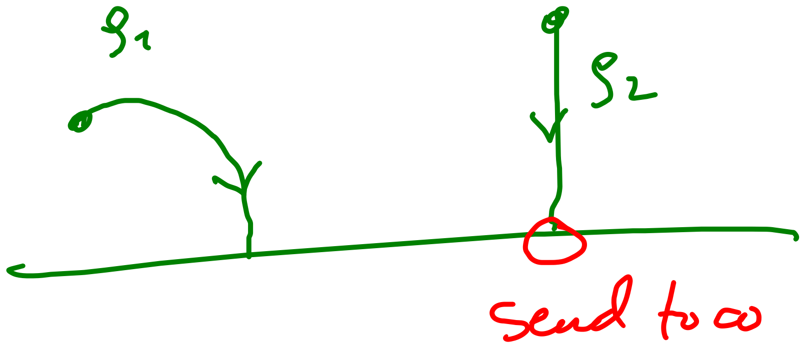
(If $g \in \operatorname{Isom} \mathbb{H}^n$, $d(g\circ\rho_1(t), g\circ\rho_2(t)) = d(\rho_1(t), \rho_2(t))$.)

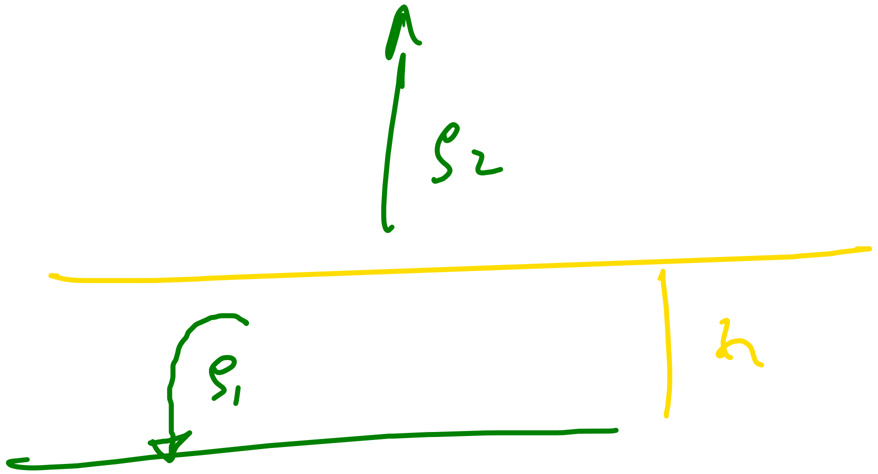
$$\cosh d(\rho_1(t), \rho_2(t)) = \frac{x_n^2 + y_n^2}{2x_n y_n} + \frac{\|\bar{x}-\bar{y}\|}{2x_n y_n} \underbrace{e^{-2t}}_{\substack{\to 0 \\ t\to\infty}}$$

bounded $\Rightarrow \rho_1 \sim \rho_2$

$$\rho_1(t) = (\bar{x}, x_n e^t)$$
$$\rho_2(t) = (\bar{y}, y_n e^t)$$

Assume ρ_1, ρ_2 have different endpts. $\rightsquigarrow$ Assume endpt of ρ_1 is ∞, ρ_2 is 0.



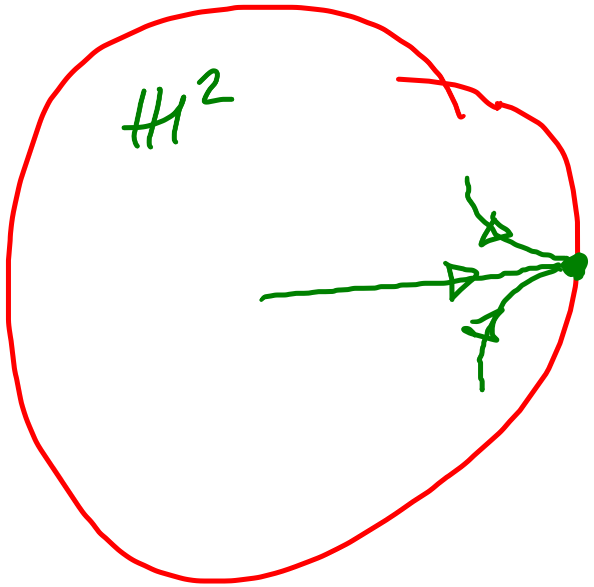


$\boxed{\rho_1 \not\sim \rho_2.}$

Image of ρ_i is contained under level h

$$\rho_2(t) = (\bar{x}, \underbrace{x_n e^t}_{\to\infty})$$

$$d(\rho_1(t), \rho_2(t)) \xrightarrow[t\to\infty]{} \infty$$

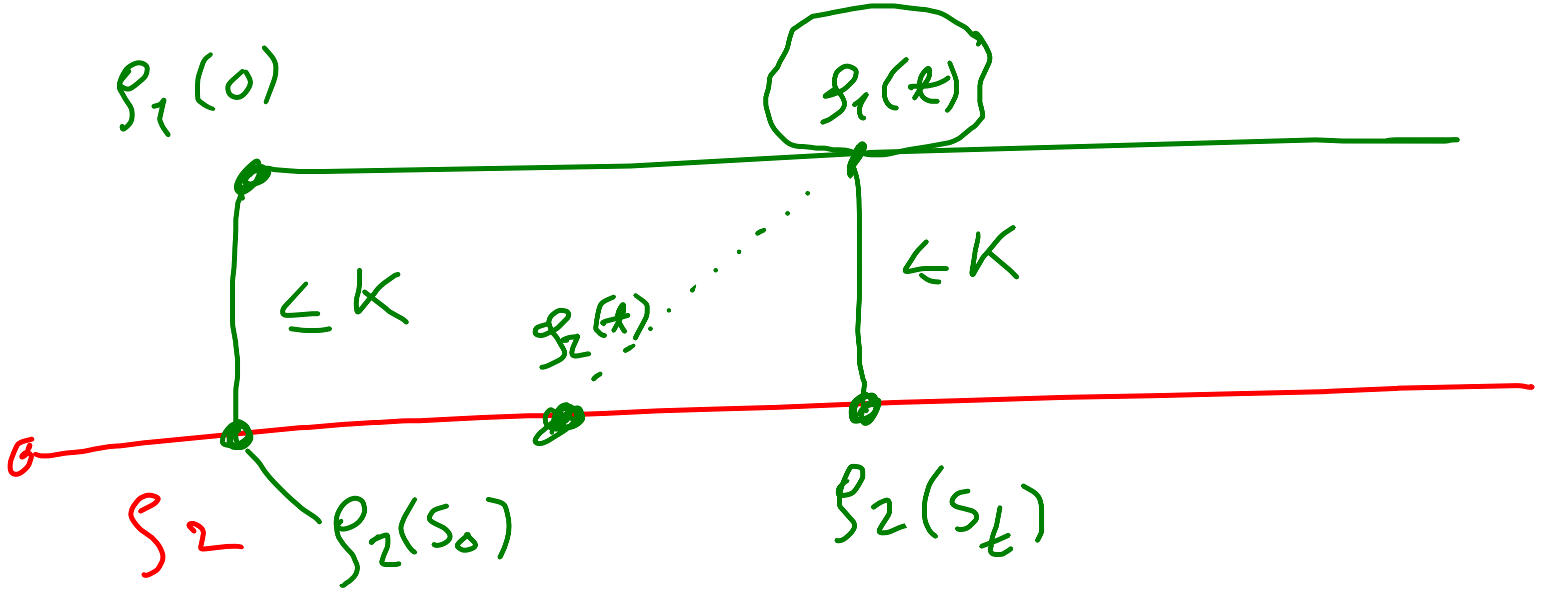


<u>Prop. 8.3</u> $\rho_1, \rho_2 \in \mathcal{G}_+(X)$. $\rho_1 \sim \rho_2 \Leftrightarrow d_{Haus}(\rho_1([0,\infty[), \rho_2([0,\infty[)) < \infty$.

<u>Proof</u>. $\rho_1 \sim \rho_2 \Rightarrow$ finite Hausdorff dist of images ok.

$\underbrace{d_{Haus}(\rho_1, \rho_2) < K.}_{t}$ $\quad t \geq 0 \Rightarrow \exists\, s_t \geq 0$ s.t. $d(\rho_1(t), \rho_2(s_t)) \leq K.$ $\quad t$

$$d(\rho_1(t), \rho_1(0)) - 2K \leq \overbrace{d(\rho_2(s_t), \rho_2(s_0))}^{|s_t - s_0|} \leq d(\rho_1(t), \rho_1(0)) + 2K$$



ρ_1, ρ_2 isom. embeddings.

$\Rightarrow t - 2K \leq |s_t - s_0| \leq t + 2K$

$$d(\rho_1(t), \rho_2(t)) \leq \underbrace{d(\rho_1(t), \rho_2(s_t))}_{\leq K} + \underbrace{d(\rho_2(s_t), \rho_2(t))}_{s_0 + 2K} \leq s_0 + 3K.$$

$\square$