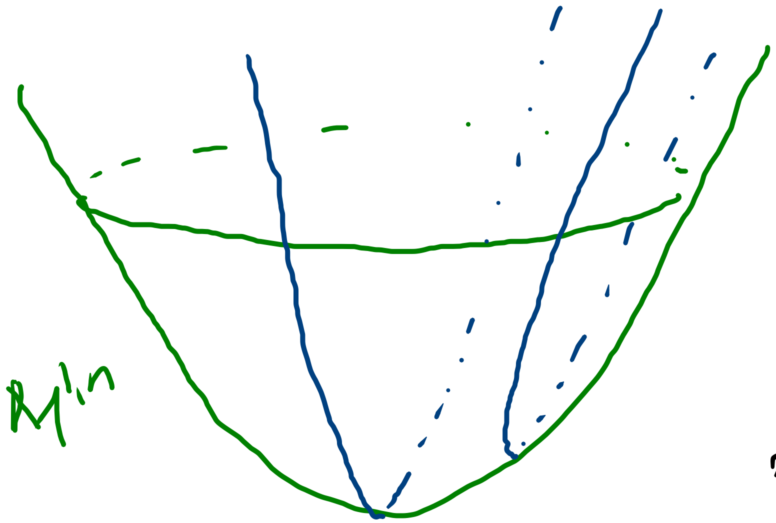
Neg. curved spaces 30.9.2020

$\mathbb{B}^n \subset \mathbb{E}^n$

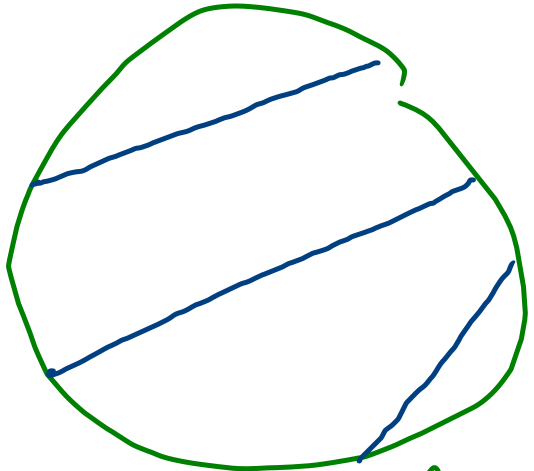
K, P, h, F isometries.



$\mathbb{H}^n \subset \mathbb{M}^{1,n}$

hyperboloid model

$\xrightarrow{K}$



Klein model

if $a, b \in \partial\mathbb{B}^n$, $a \neq b$ then $\exists_1$ geod. line with endpts a, b in $\overline{\mathbb{B}^n}$

$\boxed{\text{Prop. 5.7}}$

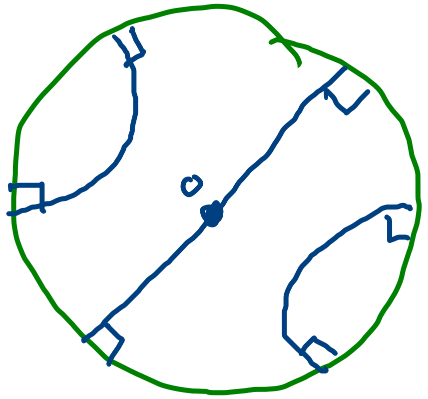
$\boxed{h = P \circ K^{-1}}$

restriction of a map defined $\overline{\mathbb{B}^n} \to \overline{\mathbb{B}^n}$

$h|_{\partial\mathbb{B}^n} = id$.

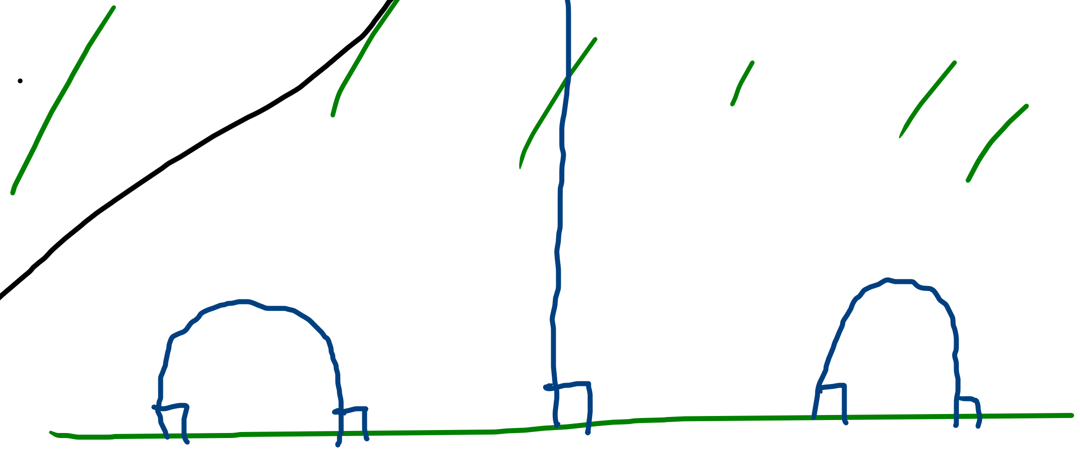
$\downarrow P$

$\mathbb{B}^n \subset \mathbb{E}^n$

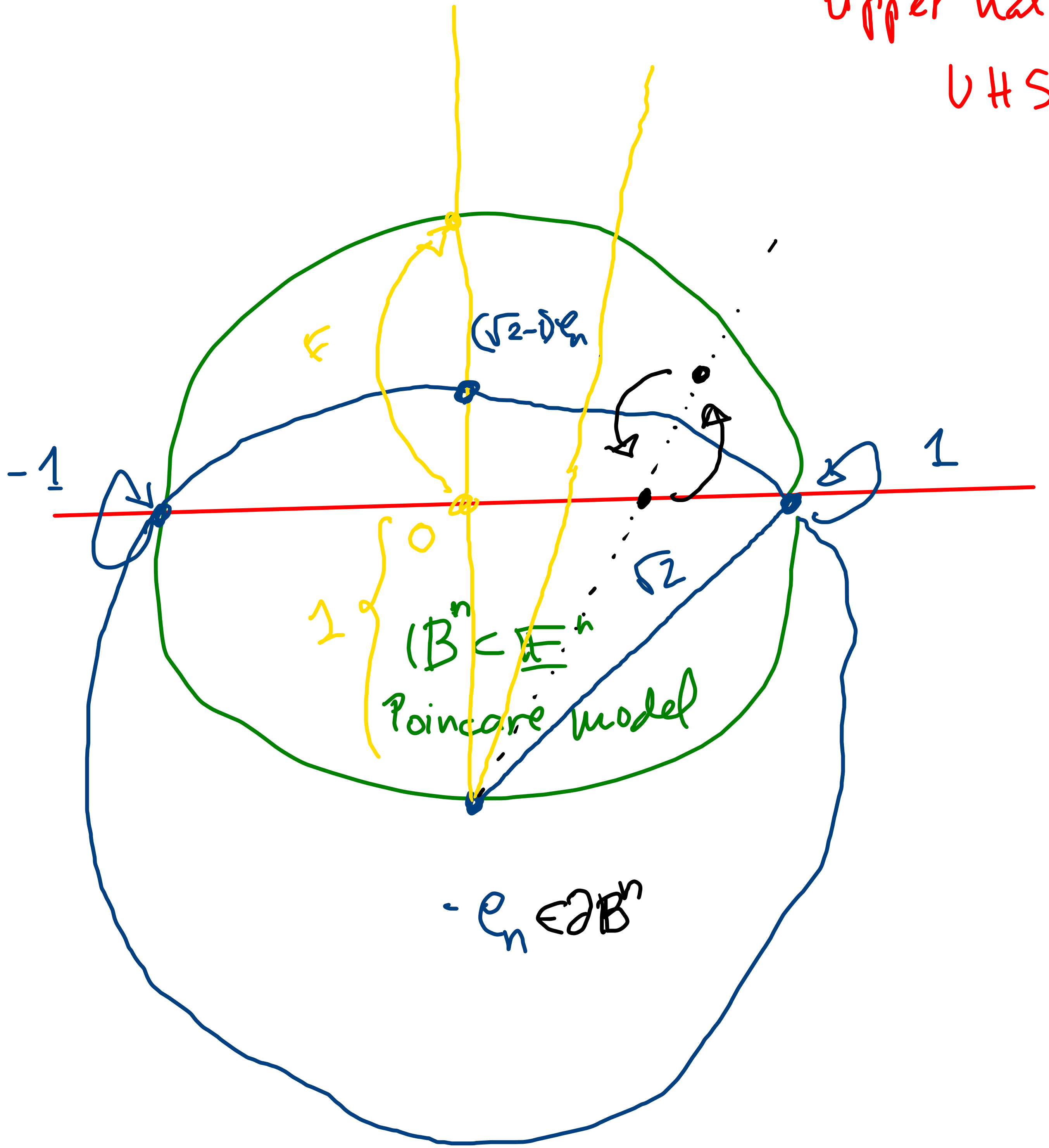


Poincaré model

$\xrightarrow{F}$



upper halfspace $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \mathbb{R}_+$



upper halfspace
UHS

$$\|x+e_n\|\,\|F(x)+e_n\| = 2.$$

$$2\frac{\|x+e_n\|}{\|x+e_n\|^2}$$

$$F = I_{-e_n,2}$$

$$\boxed{F(x) = -e_n + 2\frac{x+e_n}{\|x+e_n\|^2}}$$

$-e_n \mapsto \infty$

$\infty \mapsto -e_n$

in $\hat{E}^n$

one-pt. compactification

if $\|x+e_n\| = \sqrt{2}$, then

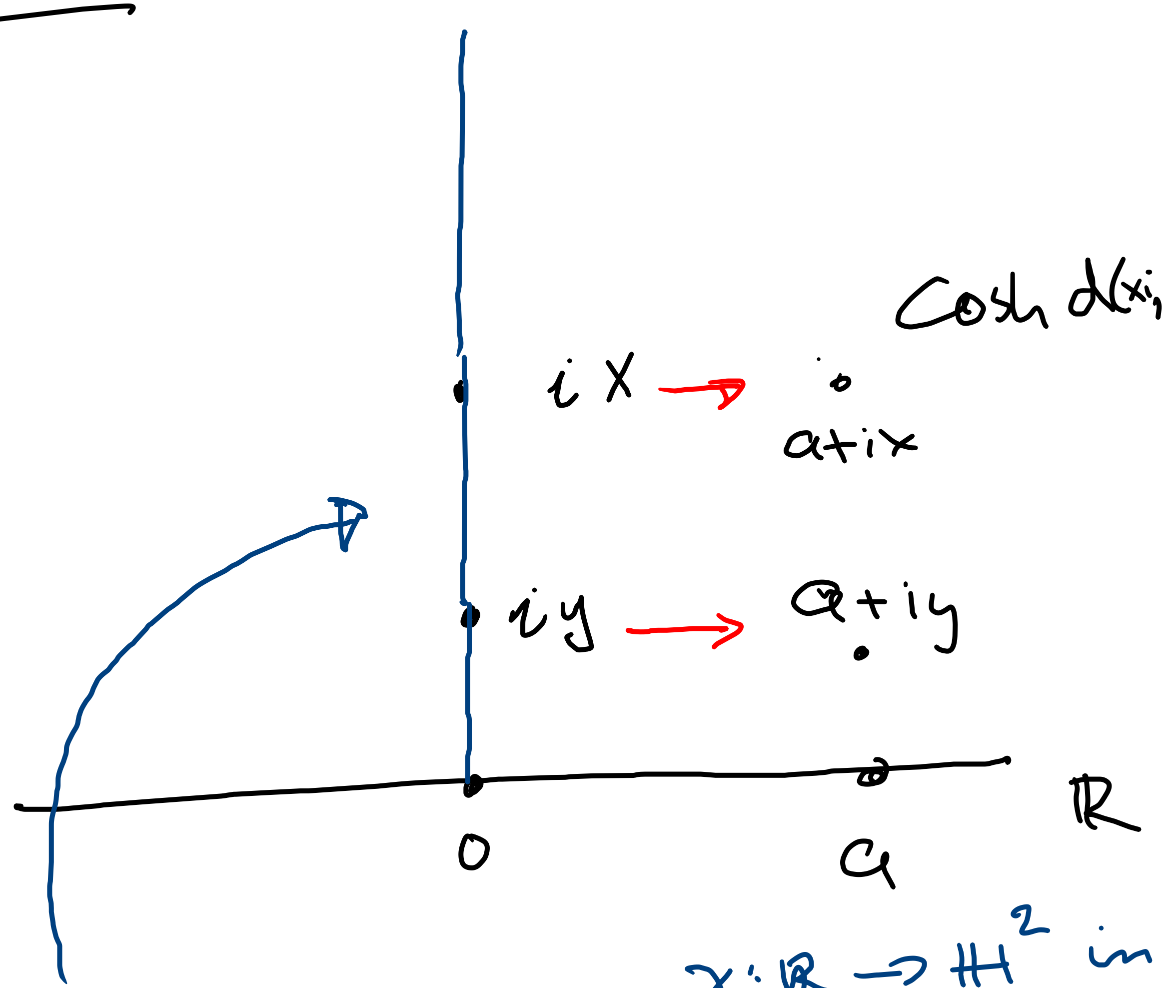
$$F(x) = -e_n + \cancel{2}\frac{x+e_n}{\cancel{2}} = x.$$

$$\boxed{F(0) = -e_n + 2\frac{e_n}{1} = e_n}$$

$$F(e_n) = 0$$

Ex. 5.5 $(n=2)$ $\quad \mathbb{R}^2_+ = \{ z \in \mathbb{C} : \operatorname{Im} z > 0 \}$.

$$d(z,w) = \operatorname{arcosh}\left(1 + \frac{|z-w|^2}{2 \operatorname{Im} z \operatorname{Im} w}\right)$$



$$\cosh d(xi, yi) = 1 + \frac{|x-y|^2}{2xy} = \frac{\cancel{2xy} + x^2 - \cancel{2xy} + y^2}{2xy}$$

$$= \frac{1}{2}\left(\frac{x}{y} + \frac{y}{x}\right) = \underline{\cosh\left(\log \frac{x}{y}\right)}$$

$$\left(\cosh t = \frac{e^t + e^{-t}}{2}\right)$$

$$\Rightarrow \boxed{d(ix, iy) = \log \frac{x}{y}}$$

$\gamma : \mathbb{R} \to \mathbb{H}^2$ in UHS model

$\gamma(t) = i e^t$ is a geod. line:

$$\cosh d(\gamma(t), \gamma(s)) = \cosh \log \frac{e^{it}}{e^{is}}$$

$$= \cosh \log e^{i(t-s)} = \cosh(t-s)$$

$\leadsto$ horizontal translations are isometries of the UHS model

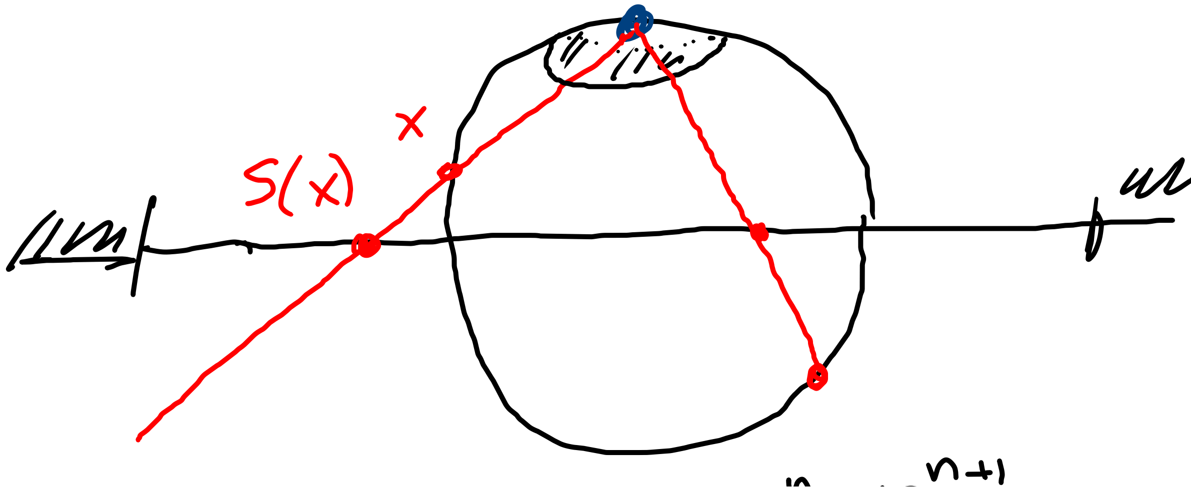
$b \in \mathbb{R}^{n-1} \times \{0\}$ $\quad T_b(x) = x + b$.

③

Stereographic projection $S: \mathbb{S}^n - \{e_{n+1}\} \to \mathbb{E}^n$

homeo.

e_{n+1} north pole

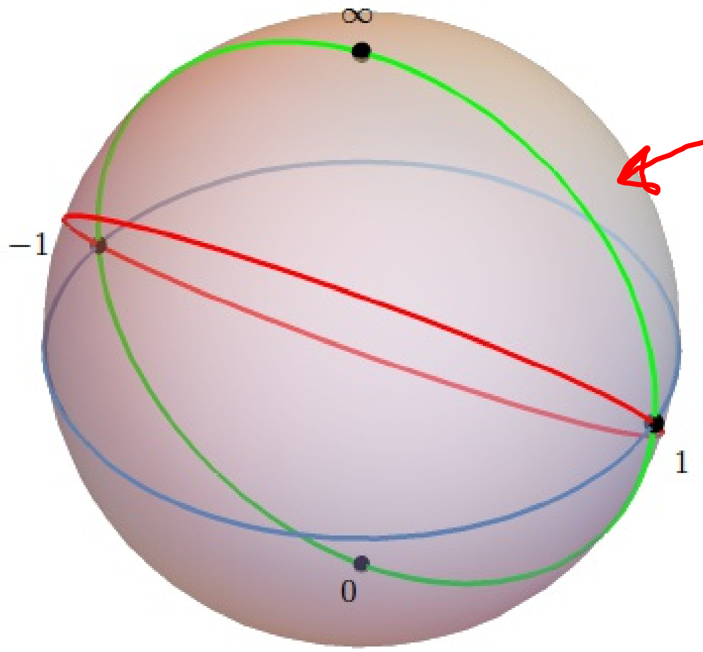


$\mathbb{E}^n \subset \mathbb{E}^{n+1}$

1-pt compactification of $\mathbb{E}^n$

$\hat{\mathbb{E}}^n = \mathbb{E}^n \cup \{\infty\}$

homeo with $\mathbb{S}^n$ by Stereogr. proj extended by setting $e_{n+1} \mapsto \infty$



UHS

$F: \mathbb{B}^n \to UHS$ is reflection in the red "spherical hyperspace"

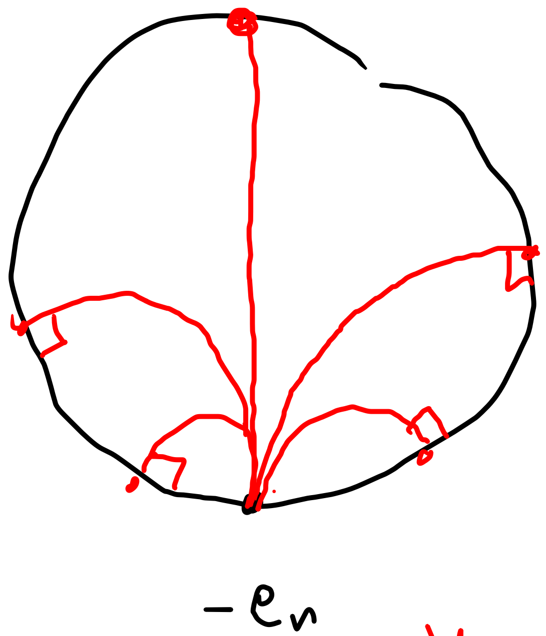
southern hemisphere $= \mathbb{B}^n$

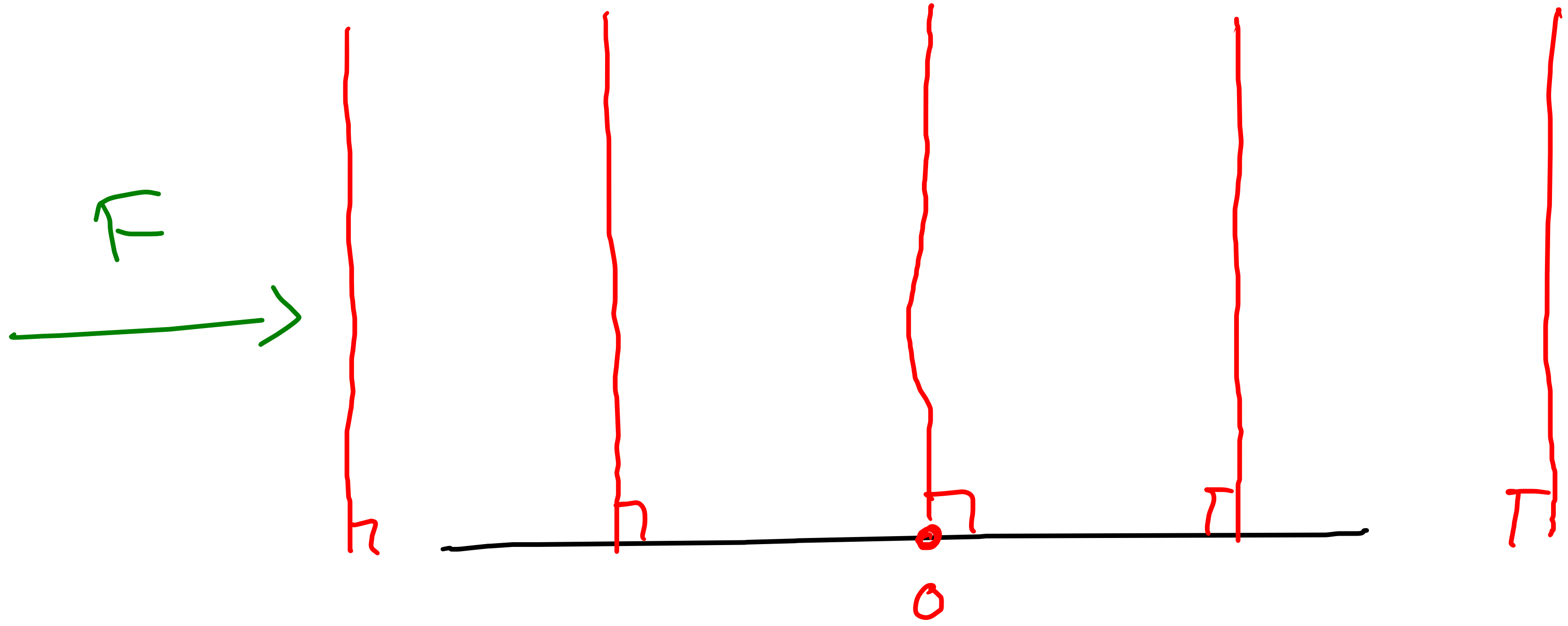
north pole $= \infty$

south pole $= 0$

F is the restriction of a homeo $\hat{\mathbb{E}}^n \circlearrowleft$

Poincaré model





UHS

geod. lines with endpt $-e_n$

geod. lines with endpoint $\infty \in \partial_\infty \mathbb{H}^n \subset \hat{\mathbb{E}}^n$ in the UHS model

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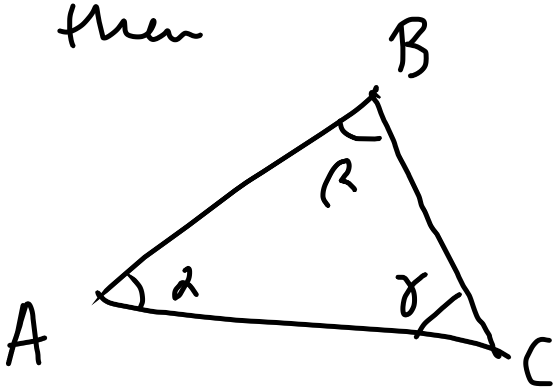
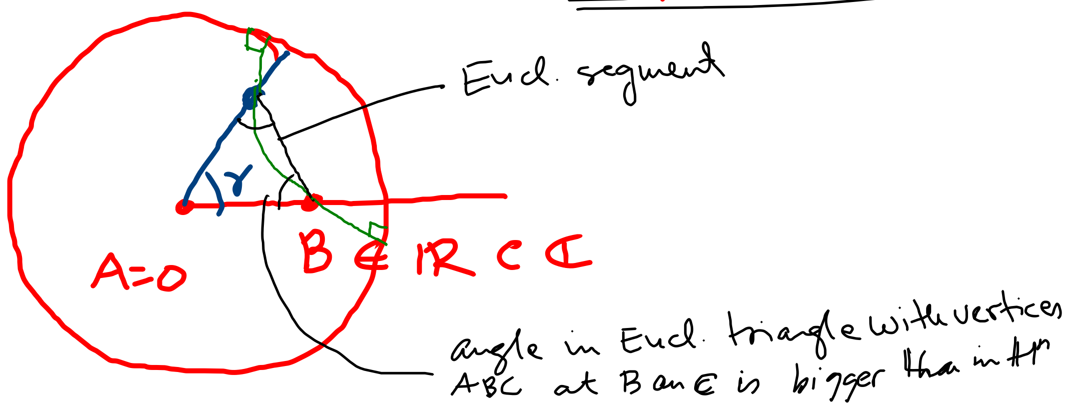
Prop. 5.10 1) if $A, B, C \in \mathbb{H}^n$ are vertices of a nondegenerate triangle then

$$\alpha + \beta + \gamma < \pi \qquad (\text{in } \mathbb{E}^n \quad \underline{\underline{\alpha+\beta+\gamma = \pi}})$$



$\mathbb{H}^2$



Poincaré

will be given an abstract defn in connection with Gromov hyp. spaces