Neg. curved spaces 29.10.2020

G group, S symm. generating set.

Word metric: $d_S(g,h) = \min\{n \in \mathbb{N} : g^{-1}h = s_1 \cdots s_n,\ s_i \in S\}$.

Cayley graph $\mathcal{G}(G,S)$: Vertices $= G$, Edges $= G \times S$

$$g \xrightarrow[(g,s)]{\text{length } 1} gs$$

Metric of Cayley graph $\mathcal{G}(G,S)$ restricts to the word metric d_S on G.

Lemma: If S, T are finite symm. gen. sets of G, then (G, d_S) and (G, d_T) are quasi-isom.

Lemma (G, d_S) and $\mathcal{G}(G,S)$ quasi-isom.

Prop. $\mathcal{G}(G,S)$ and $\mathcal{G}(G,T)$ —"—.

Def: G is a *hyperbolic group* if $\mathcal{G}(G,S)$ is hyp. for some S.

$A \neq \emptyset$, G group. G acts on A if $\exists$ homomorphism $h : G \to$ Permutation of A $= S(A)$

$g_1, g_2 \in G$ $\quad \boxed{\big(h(g_1 g_2)\big)(a) = \big(h(g_1) \cdot h(g_2)\big)(a)}$ $\quad$ h — $\boxed{\text{action}}$

X top. space, G acts on X by homeom. if it acts on X and $h(G) \subset Homeo(X)$ $\quad G \curvearrowright X$

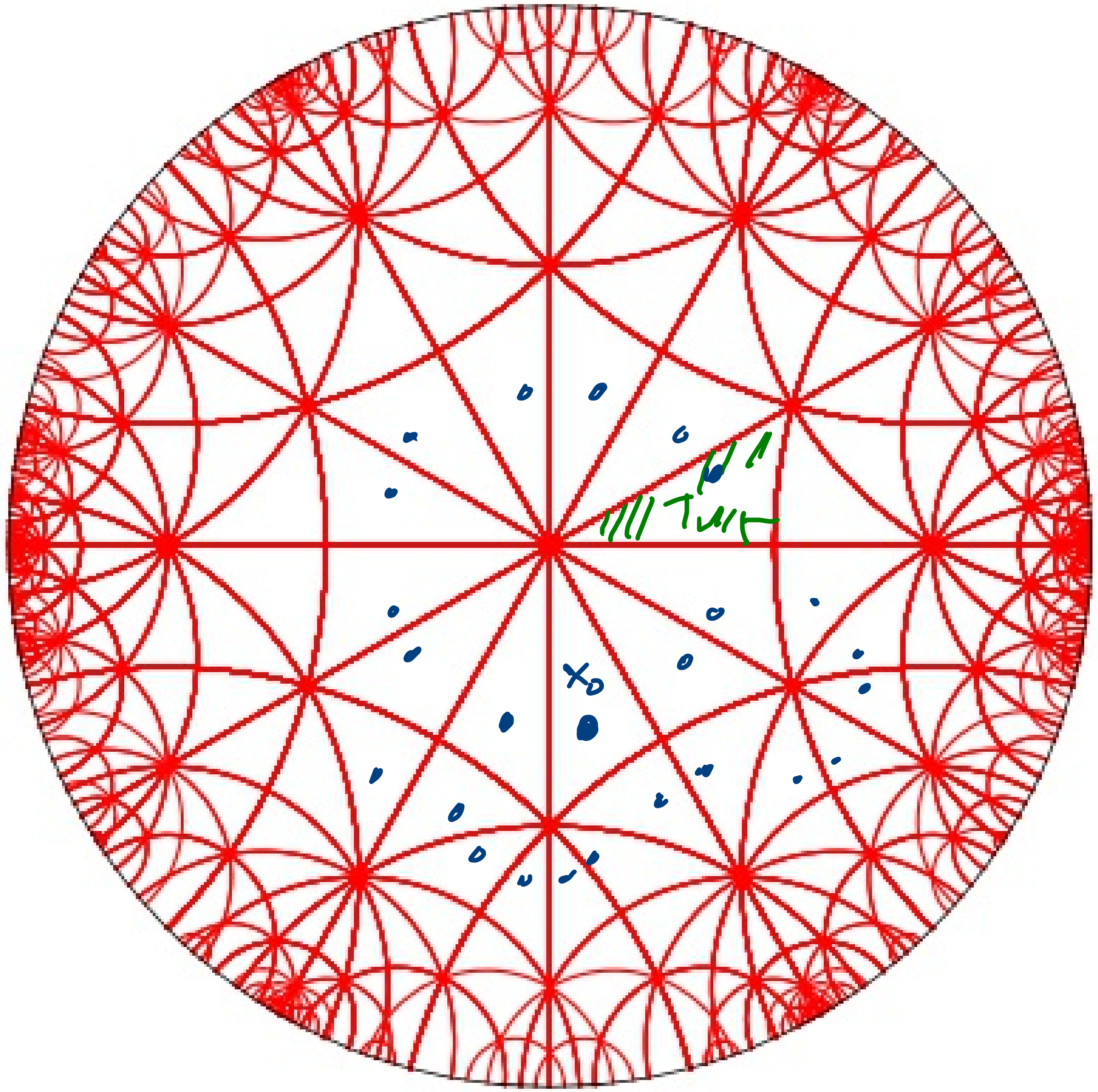
(X,d) metric space, $\boxed{G \text{ acts on } X}$ by isometries if it acts on X and $h(G) \subset Isom(X,d)$.

$G \curvearrowright (G, d_S)$ by isometries. Action: $g : G \to Isom(G, d_S)$
$g(g) = \ell_g$,
$\ell_g(g_1) = g g_1$.

Notation: $\big(h(g)\big)(x) = h(g)(x) = g \cdot x$

Example G group, S finite symm. gen. set. $g, g_1, g_2 \in G$.

$d_S(g g_1, g g_2) = \min\{n \in \mathbb{N} : g_1^{-1} \cancel{g^{-1} g}\, g_2 = s_1 \cdots s_n\} = d_S(g_1, g_2)$



<u>Example</u> $T \subset \mathbb{H}^2$ triangle with angles $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{6}$.

$\Delta(2,4,6)$ = group generated by reflections in the sides of T.

<u>Fact</u>: Images of T under $\Delta = \Delta(2,4,6)$ tile $\mathbb{H}^2$.

$x_0 \in \mathbb{H}^2$ $\quad \Delta \cdot x_0 = \{ g \cdot x_0 : g \in \Delta \} \subset \mathbb{H}^2$

discrete metric space with induced metric. Is this metric related with the word metric d_S for some finite symm. gen. set of Δ.

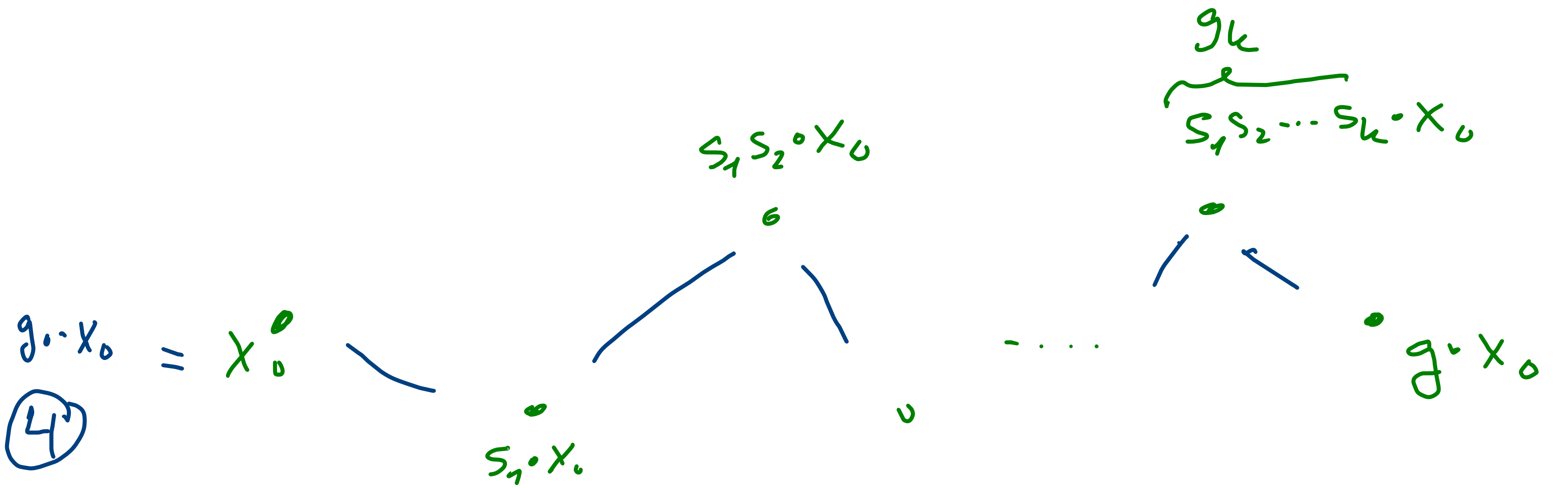
Lemma 7.20 G finitely generated group ($\exists$ finite gen. set), $G \curvearrowright X$ by isom. $x_0 \in X$.

S finite symm. gen. set of G. $\exists\, M = M(S)$ s.t.

$$d(g_1 \cdot x_0, g_2 \cdot x_0) \le M\, d_S(g_1, g_2) \quad \forall g_1, g_2 \in G$$

Proof. $M = \max\{ d(x_0, s \cdot x_0) : s \in S \}$.

$g \in G$, $g = s_1 s_2 \cdots s_n$, $g_k = s_1 \cdots s_k$, $1 \le k \le n$. $g_0 = e$.



$$d(x_0, g \cdot x_0) \le \sum_{k=1}^{n} \underbrace{d(g_{k-1} \cdot x_0, g_k \cdot x_0)}_{d(x_0,\, s_k \cdot x_0) \,\le\, M} \le nM$$

$g_1^{-1} g$

$\square$

Want: (G, d_S) quasi-isometric with $\underbrace{G\cdot x_0 = \{g\cdot x_0 : g \in G\}}_{\text{G-orbit of } x_0.}$

Need more assumptions on the action.

If $x \in X$, then $G\cdot x = G\cdot x_0$ or $G\cdot x \cap G\cdot x_0 = \emptyset$

$\rightsquigarrow$ orbits partition X $\rightsquigarrow$ define an equivalence relation in X. $/\sim$

$x \sim y \iff \exists\, g \in G$ s.t. $g\cdot x = y$.

The quotient set of this equivalence relation is

$$[x] = \{y \in X : y \sim x\} = G\cdot x$$

$$X/\sim \; = \{[x] : x \in X\} = \{G\cdot x : x \in X\} = G\backslash X$$

beware ~~$A \setminus B$~~ difference of sets $A - B$

$\pi : X \to G\backslash X$, $\pi(x) = G\cdot x$, canonical {map / projection}

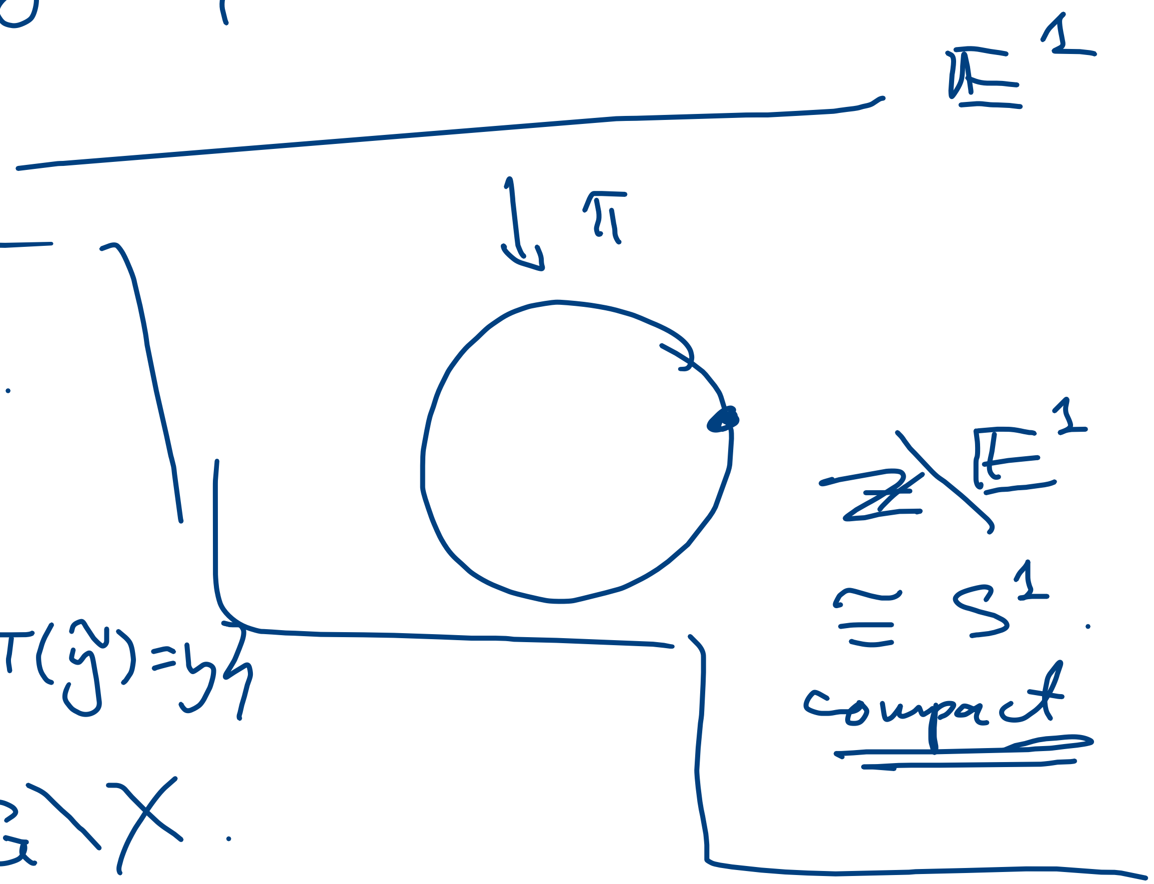
(5) The action of G is cocompact if $G\backslash X$ compact with quotient topology.

$G \curvearrowright (X,d)$ **properly** if $\cancel{\forall x \in \exists r>0}$: **NOTE: Correction here!**

$$\# \{ g \in G : \cancel{B(x,r)}^{K} \cap g \cdot \cancel{B(x,r)}^{K} \neq \emptyset \} < \infty$$ for all compact $K \subset X$

Ex. $\mathbb{Z} \curvearrowright \mathbb{E}^1$ properly cocompactly by translations

$k \cdot x = x + k$. $k \in \mathbb{Z}$, $x \in \mathbb{E}^1$.



Lemma 7.22 (X,d), $G \curvearrowright (X,d)$ by isom. properly.

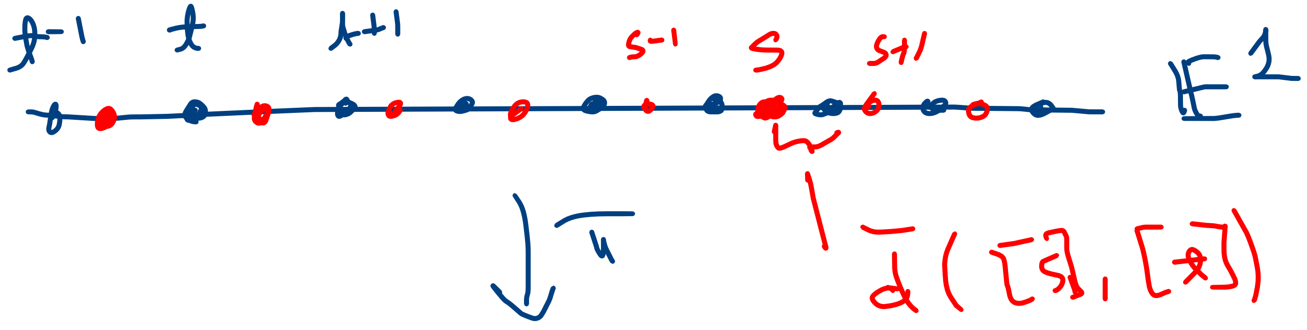
$$\bar{d}(x,y) = \min \{ d(\tilde{x}, \tilde{y}) : \pi(\tilde{x}) = x, \pi(\tilde{y}) = y \}$$

$x, y \in G \backslash X$ is a metric in $G \backslash X$.

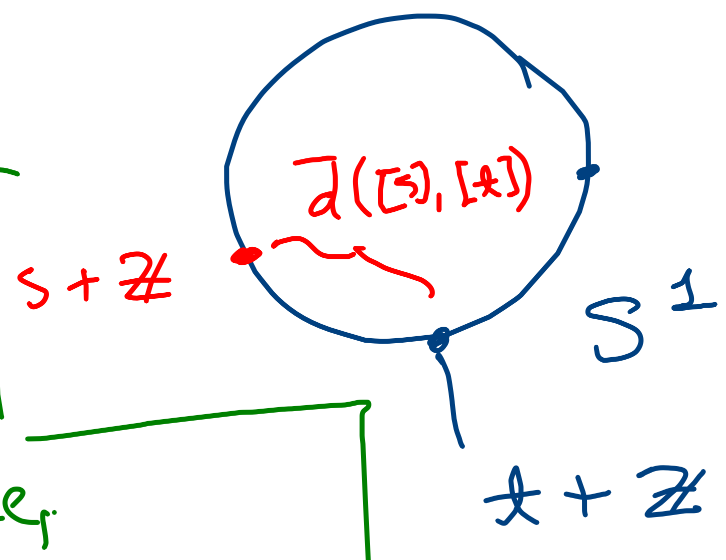
$\rightsquigarrow$ quotient metric.

Proof. Ex.

Ex. $\mathbb{Z}\backslash E^1$



$\bar{d}(\underbrace{t+\mathbb{Z}}_{[t]}, \underbrace{s+\mathbb{Z}}_{[s]})$



Thm 7.23 (Švarc, Milnor)

X proper geodesic metric space,

$G \circlearrowright X$ properly cocompactly by isom.

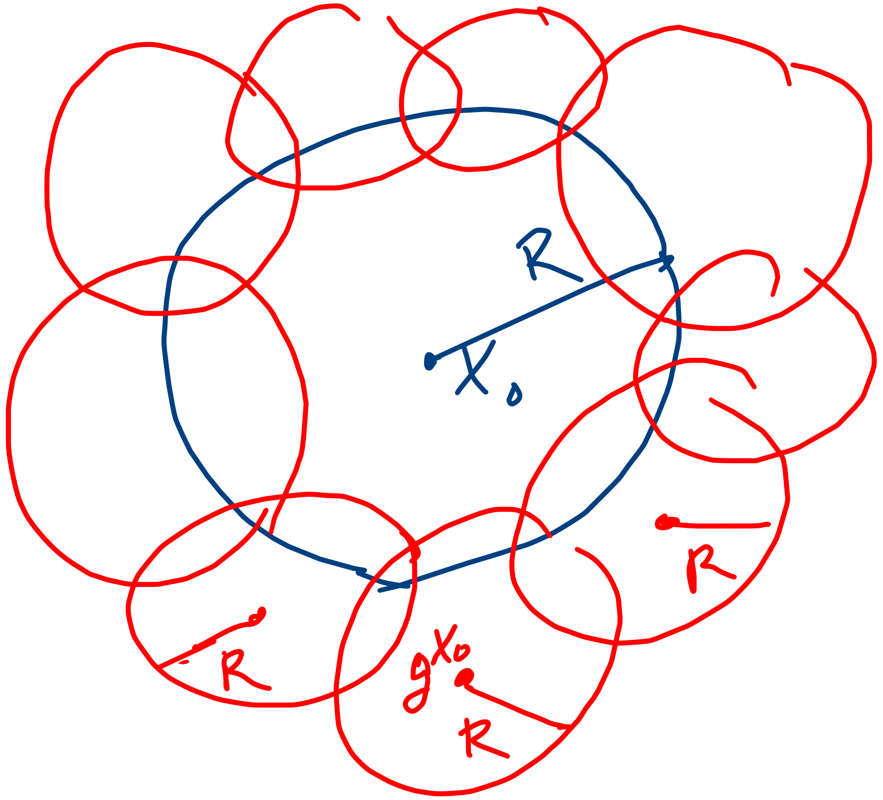
Then G is finitely generated and the map $g \mapsto g\cdot x_0$ is a quasi isometry for any $x_0 \in X$.

use word metric of any finite symm. gen. set.

$G \longrightarrow G\cdot x_0$ use the induced metric

Proof. Let $R = \operatorname{diam}(\underbrace{G\backslash X}_{\text{compact}}) < \infty$. $\quad K = \bar{B}(x_0, R)$ compact.

$\rightsquigarrow \quad X = \bigcup_{g \in G} g \cdot K.$



$S = \{ g \in G : g \cdot K \cap K \neq \emptyset \} - \{e\}$

Proper action $\Rightarrow$ S is finite.

S is symmetric:

$x \in K \cap g \cdot K$, then $g^{-1} \cdot x \in g^{-1} \cdot K \cap K$.

$g \in S \Rightarrow g^{-1} \in S$.