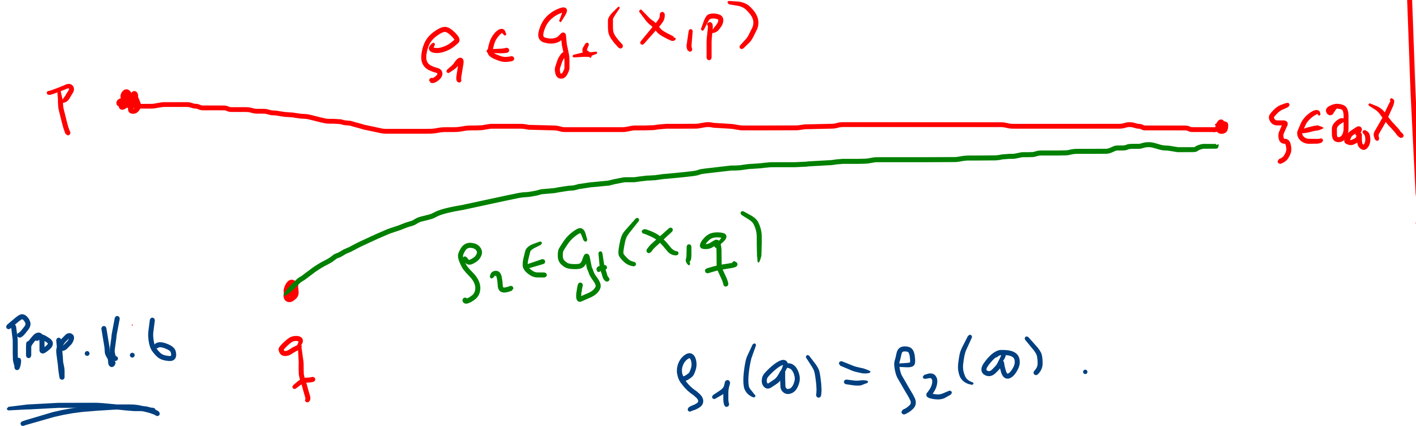
Neg. curved spaces 2.12.2020

Boundary at ∞ of CAT(-1)-spaces

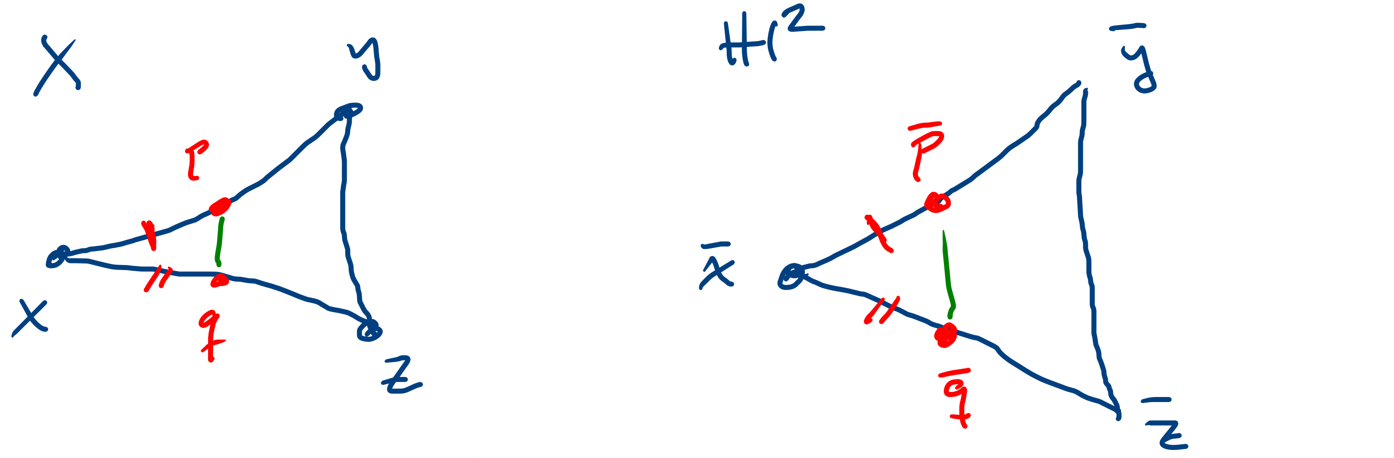


Prop. V.6

$\rho_1(\infty) = \rho_2(\infty)$.

X is CAT(-1) $\Rightarrow$ 1) $\lim_{t\to\infty} d(\rho_1(t), \rho_2) = 0$

2) $\exists T \in \mathbb{R}$ s.t. $\lim d(\rho_1(t), \rho_2(T+t)) = 0$



$d(x,y) = d(\bar{x},\bar{y})$

$d(x,z) = d(\bar{x},\bar{z})$

$d(y,z) = d(\bar{y},\bar{z})$

$d(p,q) \le d(\bar{p},\bar{q})$.

X CAT(-1) space

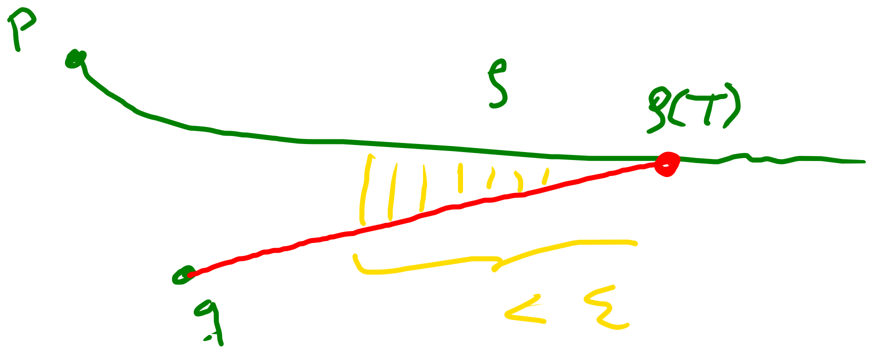
Compare with Prop. 8.4

Lemma II.3. $p,q \in \mathbb{H}^2$, $\rho \in \mathcal{G}_+(\mathbb{H}^2, p)$, $\varepsilon > 0$. If

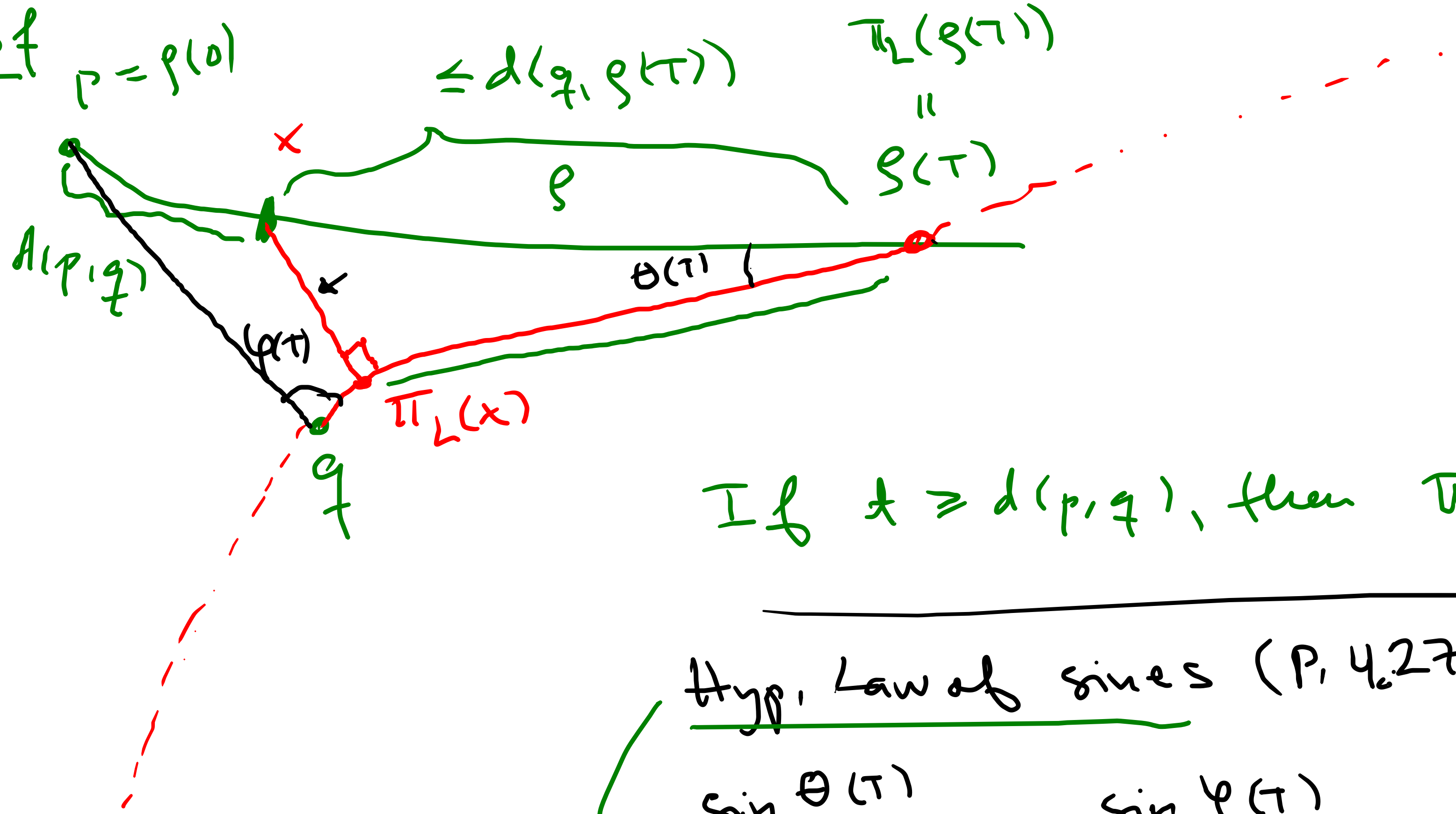
$$\max\left(d(p,q), \log \frac{4\sinh d(p,q)}{\varepsilon}\right) < t < T,$$

then $d(\rho(t), [q, \rho(T)]) < \varepsilon$.

(1)



Proof



L geod. line that contains $[q, \rho(T)]$

$\pi_L : \mathbb{H}^2 \to L$ closest point map

$$d(\pi_L(x), x) = \min_{y \in L} d(y, x).$$

Lemma: π_L is 1-Lipschitz.

If $t \geq d(p,q)$, then $\pi_L(\rho(t)) \in [q, \rho(T)]$

Lemma. $[x, \pi_L(x)]$ intersects L at a right angle.

Proof. Exercise.

Hyp. Law of sines (P. 4.27)

$$\frac{\sin\theta(T)}{\sinh d(p,q)} = \frac{\sin\varphi(T)}{\sinh T}$$

$$\Rightarrow \sin\theta(T) = \frac{\sin\varphi(T)\,\sinh d(p,q)}{\sinh T}$$

Assume $t \geq d(p,q)$

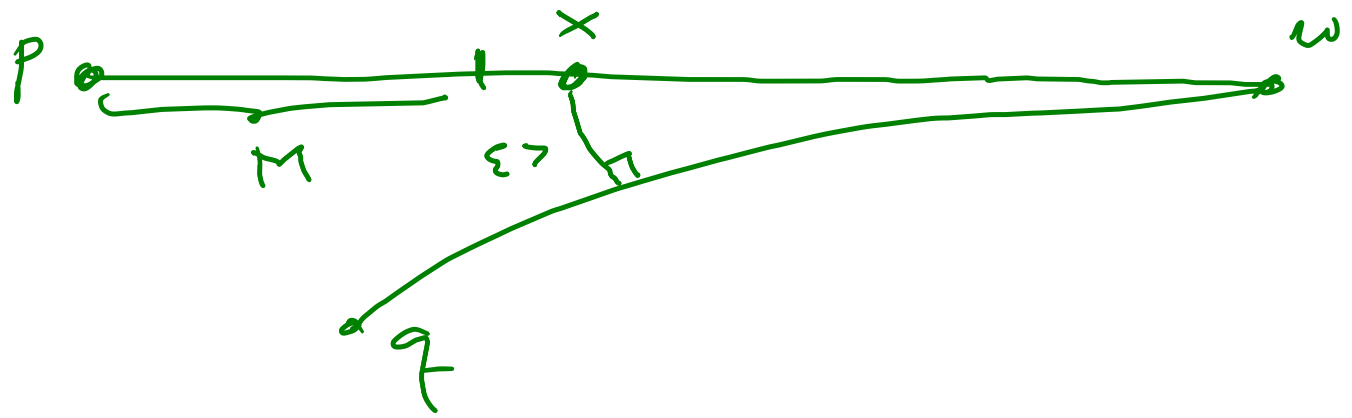
$$\sinh d(\rho(t), \pi_L(\rho(t))) = \underbrace{\frac{\sin\varphi(T)\sinh d(p,q)}{\sinh T}}_{\sin\theta(T)} \cdot \frac{\sinh(T-t)}{\sin\frac{\pi}{2} = 1} < \varepsilon$$

constant

if $t > \log \dfrac{4\sinh d(p,q)}{\varepsilon}$ (elementary). $\square$

Prop. 11.5 $\underline{X}$ CAT(-1)-space, $p, q \in X$, $\varepsilon > 0$. Then $\exists M > 0$ s.t. if $w \in X$, $d(p,w), d(q,w) \geq M$, $x \in [p,w]$ with $M \leq d(p,x)$, then $d(x, [q,w]) < \varepsilon$.

Proof. Lemma 11.3 + comparison. $\square$



Proposition 8.4. *Let X be a δ-hyperbolic space. Let ρ_1 and ρ_2 be asymptotic geodesic rays in X.*

(1) If $\rho_1(0) = \rho_2(0)$, then $d(\rho_1(t), \rho_2(t)) \leqslant 2\delta$ for all $t \geqslant 0$.

(2) For all large enough t, there is some $s_t \geqslant 0$ such that $d(\rho_1(t), \rho_2(s_t)) \leqslant 2\delta$.

(3) For all large enough t, there is some $u \in \mathbb{R}$ such that $d(\rho_1(t), \rho_2(t-u)) \leqslant 6\delta$.

proper

1) X CAT(-1), $\xi \in \partial_\infty X$, $p \in X$
$\exists_1\, \rho \in \mathcal{G}_+(X, p)$ with $\rho(\infty) = \xi$.

Proof. Existence: Prop. 9.3
Uniqueness Ex. 10.4

Prop. 11.6 is proved "in the same way", replace Rips condition by result of Prop. 11.5. $\square$

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Busemann cocycles

X CAT(-1)-space. $\beta : \partial_\infty X \times X \times X \longrightarrow \mathbb{R}$. Busemann cocycle.

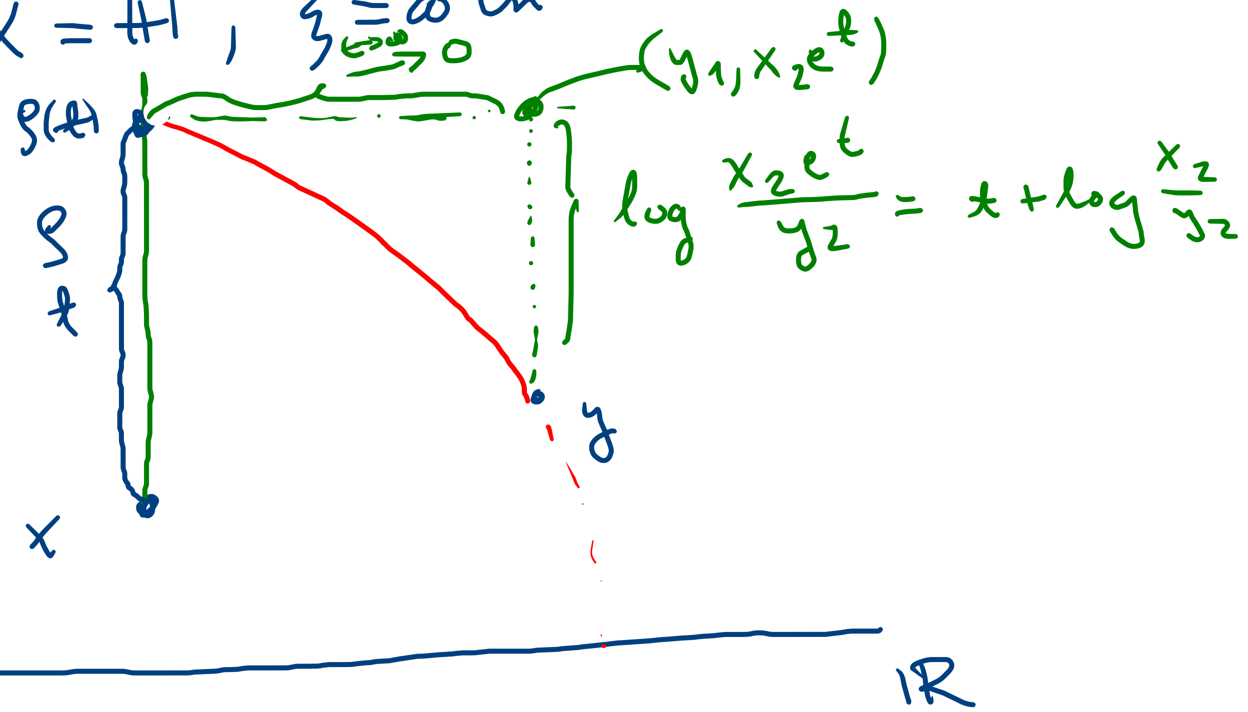
$$(\xi, x, y) \longmapsto \beta_\xi(x,y) = \lim_{t\to\infty} \big(d(\rho(t), x) - d(\rho(t), y) \big),$$

where $\rho \in \mathcal{G}_+(X)$ with $\rho(\infty) = \xi$.

Note $\beta_\xi(x,y) = -\beta_\xi(y,x)$

Prop. 11.7. β is well defined: the limit exists and it is independent of ρ.

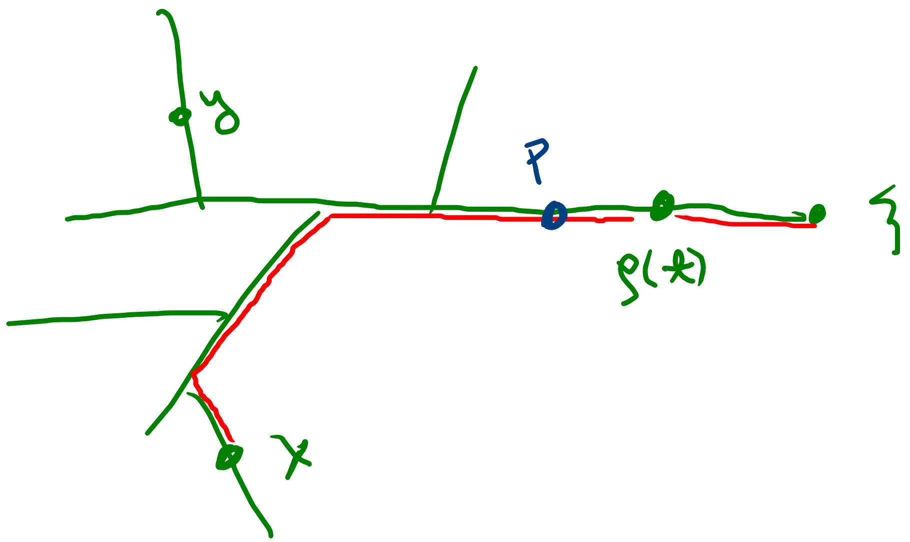
Ex. $X = \mathbb{H}^2$, $\xi = \infty$ in UHS model. Use the ray $\rho : [0,\infty[\to \mathbb{H}^2$, $\rho(t) = (x_1, x_2 e^t)$



$$t + \log\frac{x_2}{y_2} \le d(y, \rho(t)) \le t + \log\frac{x_2}{y_2} + \frac{|x_1 - y_1|}{x_2 e^t}$$

$$\Rightarrow \beta_\infty(x,y) = \lim_{t\to\infty}\big(t - d(y, \rho(t))\big) = \log\frac{y_2}{x_2}$$

2) X $\mathbb{R}$-tree, $\xi \in \partial_\infty X$;



$$\beta_\xi(x,y) = \lim_{t\to\infty} d(g(t),x) - d(g(t),y)$$

$$= d(p,x) - d(p,y)$$

for any $p \in [y,\xi[\cap [x,\xi[$.

X CAT(-1) space, $\xi \in \partial_\infty X$, $p \in X$.

$\mathcal{H}(\xi, p) = \{ y \in X : \beta_\xi(p,y) \geq 0 \}$. Horoball centered at ξ through p.

$\beta_\xi(y,p) \leq 0$

Lemma. If $g \in \mathrm{Isom}\, X$,

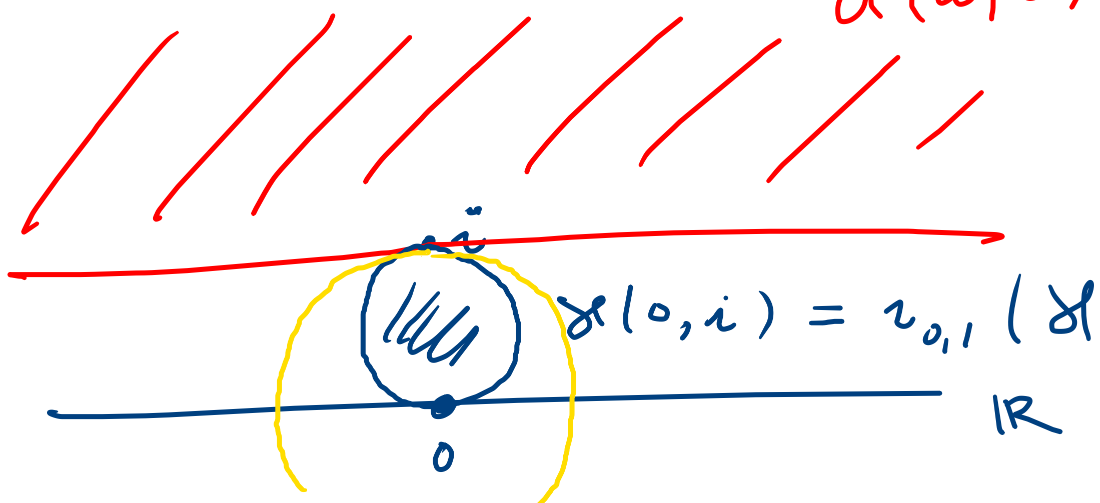
$g(\mathcal{H}(\xi,p)) = \mathcal{H}(g\cdot\xi, g(p))$

EX. $\mathbb{H}^2$ $\quad \mathcal{H}(\infty, i)$

$= \{ y \in \mathbb{H}^2 : \beta_\infty(i,y) \geq 0 \}$

$= \{ y \in \mathbb{H}^2 : \log \operatorname{Im} y \geq 0 \}$

$\iff y \geq 1 = \operatorname{Im} i$



$\mathcal{H}(0,i) = r_{0,1}(\mathcal{H}(\infty,i))$ a ball in $\mathbb{E}^2$

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