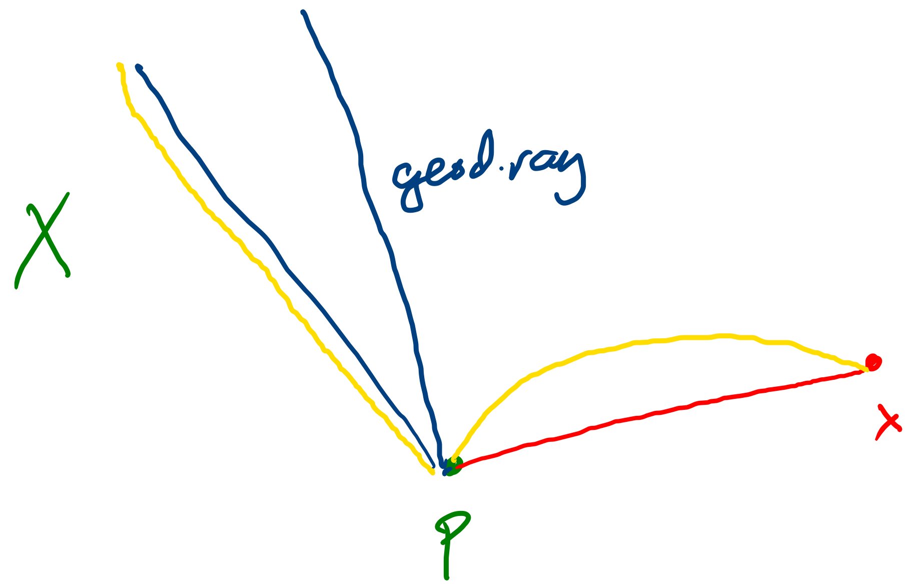
Neg. curved geometry 11.11.2020

9 Topology of $\partial_\infty X$



$\partial_\infty X = \mathcal{G}_+(X)/\sim$

$\rho_1 \sim \rho_2 \iff d_{Haus}(\rho_1, \rho_2)$ finite.

X metric space, $\omega : [0,\infty[\longrightarrow X$ is a generalized geodesic ray if ω is a geod. ray or $\exists\, m \geq 0$ s.t. $\omega|_{[0,m]}$ is a geod. segment and $\omega(t) = \omega(m)\ \forall\, t \geq m$.

(If $\gamma : [0,b] \to X$ is a geod. segment, $\check{\gamma} : [0,\infty[\to X$, $\check{\gamma}|_{[0,b]} = \gamma$, $\check{\gamma}(t) = \gamma(b)$ $\forall\, t \geq b$)

$\check{\mathcal{G}}_+(X) = \{$ gen. geod rays $\omega : [0,\infty[\to X \} \subset C([0,\infty[, X)$

$\check{\mathcal{G}}_+(X,p) = \{ \omega \in \check{\mathcal{G}}_+(X) : \omega(0) = p \} \subset C([0,\infty[, X)$

$p \in X$

topological space

use topology of compact conv.

1

Topology of compact convergence

$f_k : X \to Y$ continuous, $f_k \to f$, uniformly on compact sets

X top. space, Y metric space. if $f_k|_K \longrightarrow f|_K$ uniformly on all $K \subset X$ compact.

This defines a topology : topology of compact convergence. A basis for this topology is given by

$$B_K(f, \varepsilon) = \{ g \in C(X,Y) : \max_{x \in K} d(g(x), f(x)) < \varepsilon \}.$$

$K \subset X$ compact, $f \in C(X,Y)$, $\varepsilon > 0$.

See Munkres: Topology.

Lemma 9.1. X metric space. $p \in X$. $\check{G}_+(X)$ and $\check{G}_+(X,p)$ are closed in $C([0,\infty[, X)$.

① **Proof.** Let f be a limit point of $\check{G}_+(X)$: Let $K \subset [0,\infty[$, $n \in \mathbb{N} - \{0\}$

$B_K(f,\frac{1}{n}) \cap \check{G}_+(X) \neq \emptyset$ $\exists g_n \in \check{G}_+(X) \cap B_K(f,\frac{1}{n})$.

$\Rightarrow g_n \xrightarrow{n\to\infty} f$ uniformly on K. $\boxed{\Rightarrow \text{ Exercise}}$ $f \in \check{G}_+(X) \Rightarrow \check{G}_+(X)$ is closed.

$\check{G}(X,p)$ is closed by a similar argument. $\square$

closed balls are compact

Theorem 9.2 Let X be a proper metric space, $p \in X$. Then $\check{G}_+(X,p)$ is compact.

$\forall x_0 \in X \ \forall \varepsilon > 0 \ \exists U \ni x_0 :$
$\forall f \in \mathcal{F} \ d(f(x), f(x_0)) < \varepsilon \ \forall x \in U$.

$\check{G}_+(X,p)$

Proof. We use the Arzelà-Ascoli theorem:

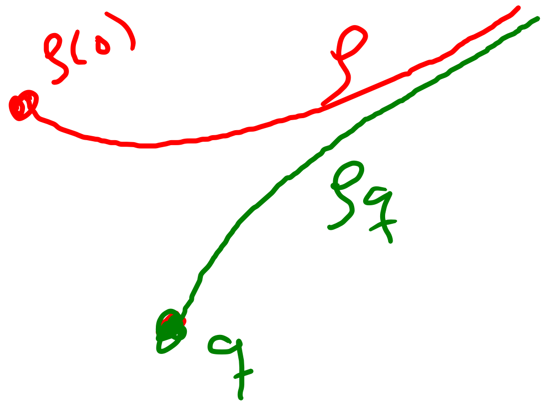
Thm B.4 Let Z $[0,\infty[$ be a top. space, X metric space. Let $\mathcal{F} \subset C(Z,X)$ be a closed subset that consists of equicontinuous mappings s.t. the sets $\mathcal{F}_z = \{f(z) : f \in \mathcal{F}\}$ have compact closures $\forall z \in Z$. Then $\mathcal{F}$ is compact.

(3) Proof. A project $\longrightarrow$ Study in Munkres' book. § 47.

$$\check{\mathcal{G}}_+(X,p)_t = \{\omega(t) : \omega \in \check{\mathcal{G}}_+(X,p)\} \subset \underbrace{\bar{B}(p,t)}_{\text{compact}}$$

$\Rightarrow$ $\check{\mathcal{G}}_+(X,p)_t$ is compact.

$\check{\mathcal{G}}_+(X,p)$ is equicontinuous (1-Lip maps)



Thm B.4, Lemma 9.1 $\Rightarrow$ $\check{\mathcal{G}}_+(X,p)$ is compact. $\square$

Recall $\partial_\infty X = \mathcal{G}_+(X)/\sim$. <u>Aim</u>: Replace $\mathcal{G}_+(X)$ by $\mathcal{G}_+(X,p)$ in the def$\underline{^n}$ of $\partial_\infty X$.

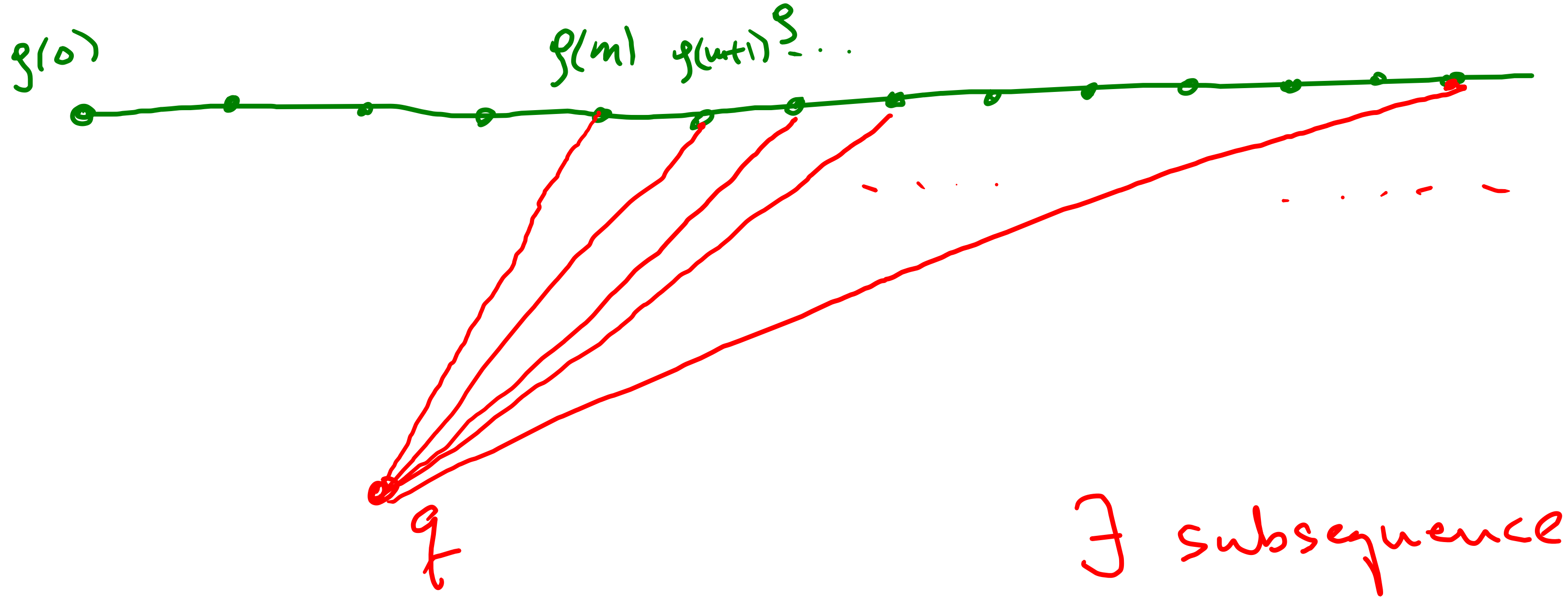
<u>Prop. 9.3</u> X proper Gromov-hyperbolic, $q \in X$. For any $\rho \in \mathcal{G}_+(X)$ there is a ray $\rho_q \in \mathcal{G}_+(X,q)$ s.t. $\rho(\infty) = \rho_q(\infty)$.

Proof: Let X be δ hyperbolic.

Let $\gamma_n : I_n \to X$, $I_n = [0, b_n]$, be geod. segments s.t.

$\gamma_n(0) = q$, $\gamma_n(b_n) = \rho(n)$.

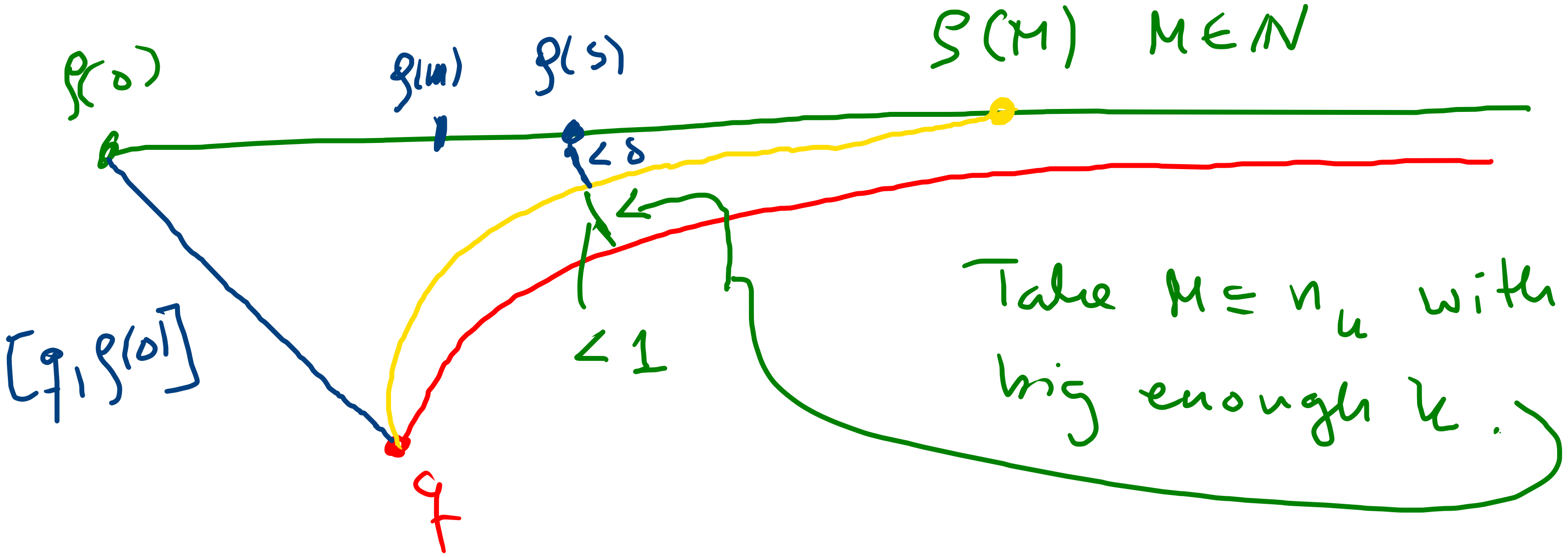
$\to$ $\gamma_n \in \mathcal{G}_+(X, q)$ compact by Thm 5.2



$\exists$ subsequence $\gamma_{n_k} \to \rho_q \leftarrow$ geod. ray.

Let us show $\rho_q(\infty) = \rho(\infty)$.

s big enough s.t. $d(\rho(s), [q, \rho(0)]) > \delta$



Take $M = n_k$ with big enough k.

$\rightsquigarrow$ $d(\rho(s), \rho_q([0,\infty[)) \le \delta + 1$

$\rightsquigarrow$ $\rho([0,\infty[) \subset \overline{W}(\rho_q([0,\infty[))$

Reverse inclusion proved in similar way.

$\square$

Cor. If X is a proper Gromov-hyperbolic space, the sets $\mathcal{G}_+(X)/\sim$ and $\mathcal{G}_+(X,q)/\sim$ are identified in a natural way.

($\exists$ "natural" bijection between these sets)

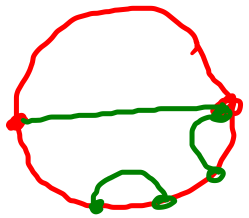
Proof. $g \in \mathcal{G}_+(X) \overset{p.9.3}{\rightsquigarrow} \exists\, g_q$ with $g(\infty) = g_q(\infty)$.

$\Rightarrow$ can take a rep. of each asympt. class $\boxed{\text{in } \mathcal{G}_+(X,q)}$ $\square$

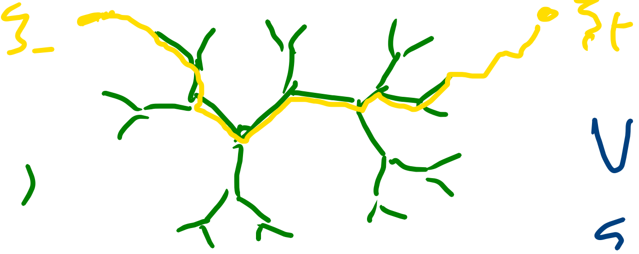
Eberlein – O'Neill 1973

Defn A geod. metric space X is a visibility space if $\forall\, \xi_-, \xi_+ \in \partial_\infty X$, $\xi_1 \neq \xi_2$, $\exists\, g \in \mathcal{G}(X)$ with $g(-\infty) = \xi_1$ and $g(+\infty) = \xi_2$.

$\uparrow$ geod. lines

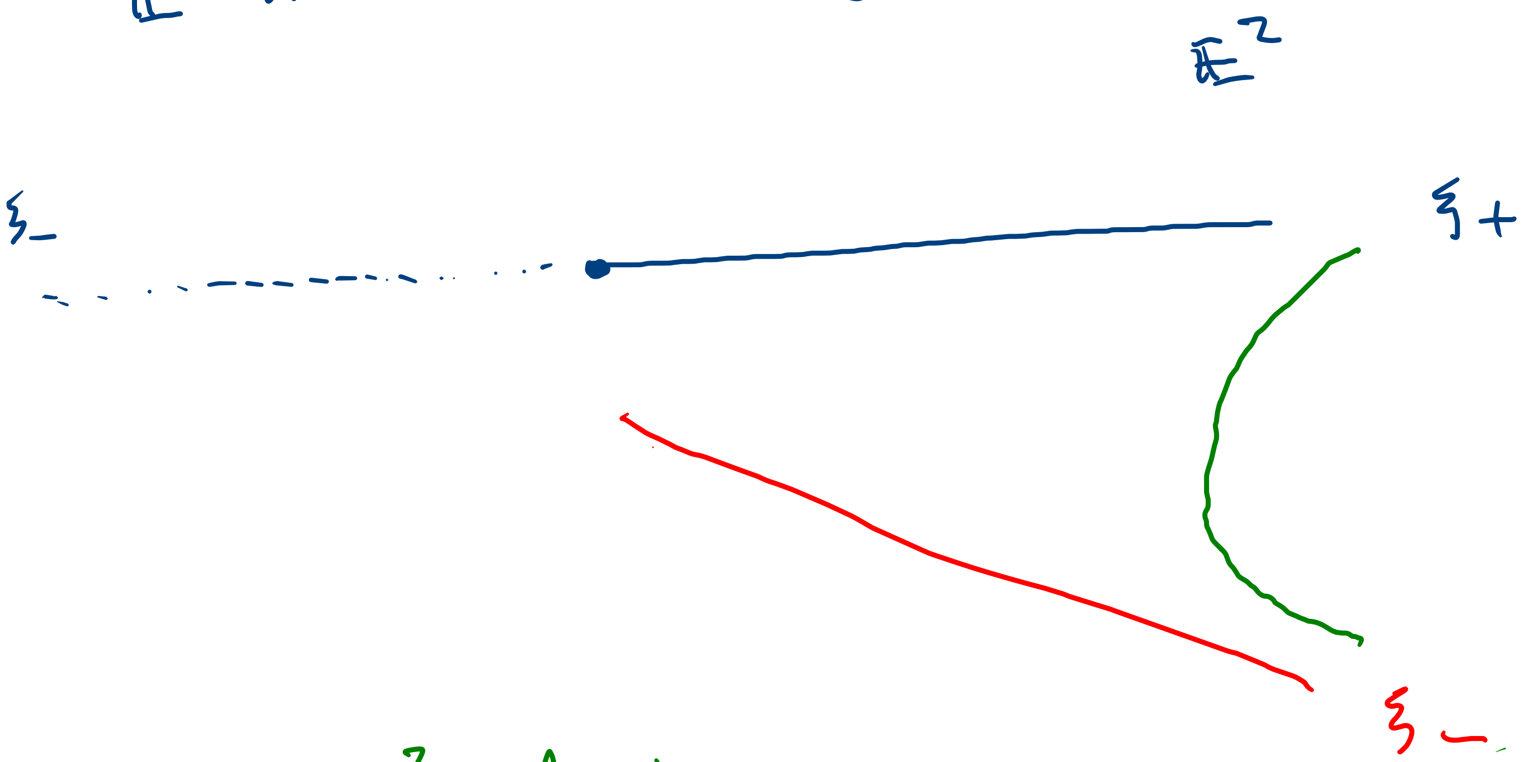
Ex. 

$\mathbb{H}^2$,



Visibility space.

$\mathbb{E}^2$ is not a visibility space.



In $\mathbb{E}^2$ if two rays are asymptotic their negative rays (contained in the lines that contain the orig. rays) are also asymptotic.

$\Rightarrow$ two points in $\partial_\infty \mathbb{E}^2$ are connected $\Leftrightarrow$ they can be represented by the ends of a single line.

7