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Exercises 8, 12.3.2018

1. Let H be a Hilbert space and let U be a closed linear subspace of H. Prove that
(a) $\ker P_U = U^\perp$,
(b) $\mathrm{id} - P_U = P_{U^\perp}$.

2. Let H be a Hilbert space and let U be a closed linear subspace of H. Prove that $(P_U x \mid y) = (x \mid P_U y)$ for all $x, y \in H$.

Let $P \in \mathrm{Lin}_b(H,H)$be an operator such that $P \circ P = P$ and $(x \mid Py) = (Px \mid y)$ for all $x, y \in H$.

3. Prove that $U = P(H)$ is a closed subspace.

4. Prove that $P = P_U$.

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5. Let $M \neq \emptyset$ be a subset of a Hilbert space. Prove that $\langle M \rangle$ is dense if and only if $M^\perp = \{0\}$.

6. Prove that an orthonormal set is linearly independent.

7. Let $\{e_1, \ldots, e_N\}$ be a finite orthonormal set and let $U = \langle e_1, \ldots, e_N \rangle$.Prove that

$$P_U v = \sum_{j=1}^{N} (v \mid e_j)\, e_j$$

for all $v \in H$.

8. Let $\mathrm{c}_n, \mathrm{s}_n \colon [0, 2\pi] \to \mathbb{R}$ be the functions

$$\mathrm{c}_n(t) = \frac{1}{\sqrt{\pi}} \cos(n\,t) \quad \text{and} \quad \mathrm{s}_n(t) = \frac{1}{\sqrt{\pi}} \sin(n\,t)\,.$$

Prove that

$$\big\{\mathrm{c}_n : n \in \mathbb{N} - \{0\}\big\} \cup \Big\{\frac{1}{\sqrt{2\pi}}\Big\} \cup \big\{\mathrm{s}_n : n \in \mathbb{N} - \{0\}\big\}$$

is an orthonormal set in $L^2([0, 2\pi])$.[1]

[1] One can get useful trigonometric identities from the equations $e^{int}e^{imt} = e^{i(n+m)t}$.