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Exercises 2, 22.1.2018

1. Let $(x_k)_{k=1}^{\infty}$ and $(y_k)_{k=1}^{\infty}$ be convergent sequences in a normed space V and let $(\lambda_k)_{k=1}^{\infty}$ and $(\mu_k)_{k=1}^{\infty}$ be convergent sequences in the field $\mathbb{K}$. Prove that

$$\lim_{k\to\infty}(\lambda_k x_k + \mu_k y_k) = \lim_{k\to\infty}\lambda_k \lim_{k\to\infty} x_k + \lim_{k\to\infty}\mu_k \lim_{k\to\infty} y_k$$

2. Prove that the closure of a vector subspace of a normed space is a vector subspace.

3. Prove that a real normed space is separable if it has a countable subset that spans a dense subspace.

Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces.

4. Prove that

$$\|(v,w)\|_\infty = \max(\|v\|_V, \|w\|_W)$$

defines a norm in $V \times W$.

5. Olkoon $1 \le p < \infty$. The expression

$$\|(v,w)\|_p = \sqrt[p]{\|v\|_V^p + \|w\|_W^p}$$

defines a norm in $V \times W$.[1] Prove that the norms $\|\cdot\|_p$ are equivalent for all $p \ge 1$ and $p = \infty$.

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6. Prove that the equivalence of norms is an equivalence relation in the set of norms of a vector space V.

7. Prove that the norms

$$\|f\|_1 = \int_{[0,1]} |f|$$

and

$$\|f\|_\infty = \max_{x\in[0,1]} |f(x)|$$

are not equivalent in $\mathrm{C}^0([0,1])$. [2]

8. Let H be a finite-dimensional subspace of a normed space V and let $x_0 \in V - H$. Prove that there is some $h_0 \in H$ for which

$$d(x_0, H) = \|x_0 - h_0\|\,.$$

[1] You are not asked to prove this!
[2] If a sequence converges in one norm and not in the other, then the norms are not equivalent.