Differential geometry 2023

Exercises 6

Let $\mathrm{Pr}^n_k \colon \mathbb{E}^n \to \mathbb{E}^k$,

$$\mathrm{Pr}^n_k(x) = \sum_{i=1}^k x^i \mathbf{e}_i \,.$$

1. Let S be a k-dimensional embedded submanifold of a smooth manifold M. Use the subspace (relative) topology in S and the atlas given by the adapted (slice) charts

$$\{(U \cap S, \phi_S) : (U, \phi) \text{ adapted chart}\}\,,$$

where $\phi_S = \mathrm{Pr}^n_k \circ \phi|_{U \cap S}$. Prove that S is a smooth k-manifold.

Solution. The sets $S \cap U$ are open in relative topology for all $U \subset M$ open. The sets $S \cap U$ cover S because the coordinate neighbourhoods U cover M. The topological space S is Hausdorff and N_2 because these properties are inherited from M. The mappings $\phi_S \colon S \cap U \to \mathrm{Pr}^n_k(\phi(U)) \subset \mathbb{E}^k$ are homeomorphisms as restrictions of homeomorphisms, so we see that S is locally Euclidean. Therefore, S with this structure is a topological manifold.

For (U, ϕ) and (V, ψ) smooth charts on M, consider the charts $(S \cap U, \phi_S)$ and $(S \cap V, \psi_S)$. Now, for $x \in \phi_S(S \cap U \cap V)$,

$$\psi_S \circ \phi_S^{-1}(x) = \mathrm{Pr}^n_k \circ \psi \circ \phi^{-1}(x, 0)$$

is smooth because it is the restriction of a smooth mapping to a coordinate plane. As ψ is a slice chart, $\psi_S \circ \phi_S^{-1}(x) \in V \cap \mathbb{E}^k$, and the mapping $\phi_S \circ \psi_S^{-1}$ is smooth as well. Thus, S is a smooth manifold.

2. Let $\tilde{f} \colon \mathbb{E}^3 \to \mathbb{E}^1$, $f(y) = y_3$ for all $y \in \mathbb{E}^3$ in the standard coordinates of $\mathbb{E}^3$. Prove that $f = \tilde{f}|_{\mathbb{S}^2} \colon \mathbb{S}^2 \to \mathbb{E}^1$ is a smooth function. Determine the critical points of f.

Solution. We check the smoothness using the (six) standard projection charts of $\mathbb{S}^2$, namely we let $k \in \{1, 2, 3\}$ and use both charts (U_k^+, pr_k) and (U_k^-, pr_k). For all points $x \in \mathrm{pr}_k(U_k^\pm) = \{\mathrm{pr}_k(y_1, y_2, y_3) \in \mathbb{E}^2 \, : \, \sum_{i \neq k} y_i^2 < 1\}$, we have

$$f \circ (\mathrm{pr}_k|_{U_k^\pm})^{-1}(x) = \begin{cases} x_3 & \text{if } k \in \{1, 2\}, \\ \pm\sqrt{1 - (x_1^2 + x_2^2)} & \text{if } k = 3. \end{cases}$$

In both cases, this is the formula of a smooth map and proves that f is smooth.

Recall that a point $p \in \mathbb{S}^2$ is called *critical* for f if df_p is not surjective, or equivalently that the Jacobian matrix of (the coordinate representative $f \circ (\mathrm{pr}_k|_{U_k^\pm})^{-1}$ of) f is not surjective. Since $\dim \mathbb{E}^1 = 1$, this Jacobian matrix is a gradient and we just need to find the points $p \in \mathbb{S}^2$ such that, for a chart $(U_k^\pm, \mathrm{pr}_k)$ containing p, we have the equality $\nabla(f \circ (\mathrm{pr}_k|_{U_k^\pm})^{-1})_{\mathrm{pr}_k|_{U_k^\pm}(p)} = 0$. We compute

$$\nabla(f \circ (\mathrm{pr}_k|_{U_k^\pm})^{-1})_{\mathrm{pr}_k|_{U_k^\pm}(p)} = \begin{cases} (0, 0, 1) & \text{if } k \in \{1, 2\}, \\ \pm\Big(\frac{p_1}{\sqrt{1-(p_1^2+p_2^2)}}, \frac{p_2}{\sqrt{1-(p_1^2+p_2^2)}}, 0\Big) & \text{if } k = 3. \end{cases}$$

With the case $k = 3$, we find that the critical points of f are $(0, 0, 1)$ and $(0, 0, -1)$.

3. Let S be an embedded submanifold of a smooth manifold M. Let $p \in S$ and let $v \in T_pM$. Prove that $v \in T_pS$ (seen as a subspace of T_pM with Proposition 5.18) if and only if there is a smooth path $\gamma\colon I \to M$, such that $\dot\gamma(0) = v$ and $\gamma(t) \in S$ for all $t \in I$.

Solution. Let $v \in T_pS$. By Proposition 3.12, there is a smooth path $\gamma\colon I \to S$ such that $\dot\gamma(0) = v$. Let $i\colon S \to M$ be the inclusion map. Then $i \circ \gamma$ is a smooth path and $(i \dot\circ \gamma)(0) = di_p\dot\gamma(0) = di_pv$.

On the other hand, if $\eta\colon I \to M$ is a smooth path such that $\eta(t) \in S$ for all $t \in I$, then $\eta = i \circ \eta_S$, where η_S is η considered as a map to S (it is a corestriction, sometimes written $\eta|^S$). Now $\dot\eta(0) = di_p\dot\eta_S(0) \in di_p(T_pS)$.

4. Let $\widetilde\nu\colon \mathbb{E}^3 \to \mathbb{E}^6$,

$$\widetilde\nu(x) = (x_1^2, x_2^2, x_3^2, \sqrt{2}x_2x_3, \sqrt{2}x_1x_3, \sqrt{2}x_1x_2)\,.$$

(1) Prove that $\widetilde\nu(\mathbb{S}^2) \subset \mathbb{S}^5$.
(2) Prove that $\widetilde\nu|_{\mathbb{S}^2}\colon \mathbb{S}^2 \to \mathbb{S}^5$ is a smooth immersion.
(3) Prove that the mapping $\nu\colon \mathbb{P}^2 \to \mathbb{P}^5$,

$$\nu([x]) = [x_1^2 : x_2^2 : x_3^2 : \sqrt{2}x_2x_3 : \sqrt{2}x_1x_3 : \sqrt{2}x_1x_2]$$

is a smooth embedding.

Solution. (1) For all $x \in \mathbb{S}^n$, we have

$$1 = 1^2 = (x_1^2 + x_2^2 + x_3^2)^2 = x_1^4 + x_2^4 + x_3^4 + 2x_1^2x_2^2 + 2x_2^2x_3^2 + 2x_3^2x_1^2 = \|\widetilde\nu(x)\|^2\,.$$

(2) The Jacobian matrix of $\widetilde\nu$ is

$$D\widetilde\nu(x) = \begin{pmatrix} 2x_1 & 0 & 0 \\ 0 & 2x_2 & 0 \\ 0 & 0 & 2x_3 \\ 0 & \sqrt{2}x_3 & \sqrt{2}x_2 \\ \sqrt{2}x_3 & 0 & \sqrt{2}x_1 \\ \sqrt{2}x_2 & \sqrt{2}x_1 & 0 \end{pmatrix}.$$

If x_1, x_2, x_3 are all nonzero, then

$$\det\begin{pmatrix} 2x_1 & 0 & 0 \\ 0 & 2x_2 & 0 \\ 0 & 0 & 2x_3 \end{pmatrix} = 8x_1x_2x_3 \neq 0\,.$$

If $x_1 \neq 0$, then

$$\det\begin{pmatrix} 2x_1 & 0 & 0 \\ \sqrt{2}x_3 & 0 & \sqrt{2}x_1 \\ \sqrt{2}x_2 & \sqrt{2}x_1 & 0 \end{pmatrix} = 4x_1^3 \neq 0$$

and so on. This implies that the rank of the differential matrix is 3 outside the origin. In particular, its restriction to $T_p\mathbb{S}^2$ has rank 2 at all $p \in \mathbb{S}^2$. This implies that $\widetilde\nu$ is an immersion.

In this question, we used the general following fact: for any immersion $f : M \to N$ between smooth manifolds and for any embedded submanifold S of M, the restriction

$f_{|S} : S \to M$ is still an immersion. To see this, first use the proposition (from the lecture notes) stating that the injection $i_S : S \to M$ is an embedding hence an immersion, then for all $p \in S$, apply the chain rule and to obtain $d(f_{|S})_p = d(f \circ i_S)_p = df_p \circ d(i_S)_p$.

(3) Note that $\widetilde{\nu}(x) = \widetilde{\nu}(y)$ if and only if $x = \pm y$: Clearly, $\widetilde{\nu}(-x) = \widetilde{\nu}(x)$ for all $x \in \mathbb{S}^2$. Equality of the first three components implies that there are constants $u_1, u_2, u_3 \in \{-1, 1\}$ such that $x_1 = u_1 y_1$, $x_2 = u_2 y_2$ and $x_3 = u_3 y_3$. The three last equations imply that $u_1 = u_2 = u_3$. This implies that $\widetilde{\nu}$ is compatible with the equivalence relation[1] used in the definition of $\mathbb{P}^2$, and that the mapping ν is injective.

As $\mathbb{P}^2$ is compact, and ν is injective, Proposition 5.10 implies that it suffices to show that ν is a smooth immersion. Let $q \in \mathbb{P}^2$. The mapping $\pi\colon \mathbb{S}^2 \to \mathbb{P}^2$, $\pi(x) = [x]$, is a local diffeomorphism so there is an open neighbourhood $V \ni q$ such that π has a smooth local inverse $\breve{\pi}\colon V \to \mathbb{S}^2$ that satisfies $\pi \circ \breve{\pi} = \mathrm{id}_V$. As $\nu = \widetilde{\nu} \circ \breve{\pi}$, the differential $d\nu_q = d\widetilde{\nu}_{\breve{\pi}(q)} d\breve{\pi}_q$ is injective as a composition of two injective linear mappings. This implies that ν is a smooth immersion.

5. Let X_1, X_2, X_3 be vector fields in $\mathbb{E}^4$,

$$
\begin{aligned}
X_1(x) &= -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2} + x^4 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^4}\,, \\
X_2(x) &= -x^3 \frac{\partial}{\partial x^1} - x^4 \frac{\partial}{\partial x^2} + x^1 \frac{\partial}{\partial x^3} + x^2 \frac{\partial}{\partial x^4}\,, \\
X_3(x) &= -x^4 \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^3} + x^1 \frac{\partial}{\partial x^4}\,.
\end{aligned}
$$

(1) Prove that X_1, X_2, X_3 define smooth vector fields on the regular submanifold $\mathbb{S}^3 \subset \mathbb{E}^4$.

(2) Prove that $X_1(p), X_2(p), X_3(p)$ are linearly independent for all $p \in \mathbb{S}^3$.

(3) Prove that $T\mathbb{S}^3$ is diffeomorphic with $\mathbb{S}^3 \times \mathbb{E}^3$.

Solution. (1) As $\mathbb{S}^3$ is a smooth submanifold, Proposition 5.3(1) implies that the mappings $X_1|_{\mathbb{S}^3}$, $X_2|_{\mathbb{S}^3}$ and $X_3|_{\mathbb{S}^3}$ are smooth. Note that

$$
\begin{aligned}
\big(x \,|\, (-x^2, x^1, x^4, -x^3)\big) &= -x^1x^2 + x^2x^1 + x^3x^4 - x^4x^3 = 0\,, \\
\big(x \,|\, (-x^3, -x^4, x^1, x^2)\big) &= -x^1x^3 - x^2x^4 + x^3x^1 + x^4x^2 = 0 \quad \text{and} \\
\big(x \,|\, (-x^4, x^3, -x^2, x^1)\big) &= -x^1x^4 + x^2x^3 - x^3x^2 + x^4x^1 = 0\,.
\end{aligned}
$$

Thus, $X_1|_{\mathbb{S}^3}(x), X_2|_{\mathbb{S}^3}(x), X_3|_{\mathbb{S}^3}(x) \in T_x\mathbb{S}^n$ for all $x \in \mathbb{S}^n$. Example 5.20 implies that $X_1|_{\mathbb{S}^3}, X_2|_{\mathbb{S}^3}, X_3|_{\mathbb{S}^3} \in \mathfrak{X}(\mathbb{S}^n)$.

(2) A computation shows that the four 3×3-subdeterminants of the coefficients of the three vector fields are $-x^4\|x\|^2$, $-x^2\|x\|^2$, $x^3\|x\|^2$ and $x^1\|x\|^2$. If $x \neq 0$, then at least one of these is nonzero, and we conclude that the vector fields are linearly independent at all $x \in \mathbb{S}^3$.

(3) This follows from the claim of Problem 2 of Exercises 5.

[1] $x \sim y$ if and only if $y = \pm x$