Differential geometry 2023

Exercises 4

1. (1) Let $F\colon M \to N$ be a smooth mapping and let $p \in M$. Prove that dF_p is a linear mapping.
(2) Let M_1, M_2 and M_3 be smooth manifolds and let $F_1\colon M_1 \to M_2$ and $F_2\colon M_2 \to M_3$ be smooth mappings. Prove that $d(F_2 \circ F_1)_p = (dF_2)_{F_1(p)}(dF_1)_p$.

Solution. (1) Let $v, w \in T_pM$ and let $\lambda, \mu \in \mathbb{R}$. Let $f \in \mathrm{C}^\infty(N)$. Then

$$\begin{aligned} dF_p(\lambda v + \mu w)f &= (\lambda v + \mu w)(f \circ F) = \lambda v(f \circ F) + \mu w(f \circ F) \\ &= \lambda(dF_p v)f + \mu(dF_p w)f = (\lambda(dF_p v) + \mu(dF_p w))f \,. \end{aligned}$$

(2) Let $v \in T_pM_1$ and let $f \in \mathrm{C}^\infty(M_3)$. Then

$$(d(F_2)_{F_1(p)} d(F_1)_p v)f = (d(F_1)_p v)(f \circ F_2) = v(f \circ F_2 \circ F_1) = d(F_2 \circ F_1)vf \,.$$

2. (1) Prove that the differential of the identity map of a smooth manifold M at a point $p \in M$ is id_{T_pM}.
(2) Let $F\colon M \to N$ be a smooth diffeomorphism and let $p \in M$. Prove that dF_p is a linear bijection and that $(dF_p)^{-1} = (dF^{-1})_{F(p)}$.

Solution. (1) If $v \in T_pM$ and $f \in \mathrm{C}^\infty(M)$, then

$$d(\mathrm{id}_M)_p vf = vf \circ \mathrm{id}_M = vf = \mathrm{id}_{T_pM} f \,.$$

(2) This follows directly from part (2) of Exercise 1 and part (1) of this exercise.

3. The spherical coordinates of a point $x \in \mathbb{E}^3 \setminus \{0\}$ are given by

$$x = (r\cos\theta_1 \sin\theta_2, r\sin\theta_1 \sin\theta_2, r\cos\theta_2) \,.$$

Let $p \in \mathbb{E}^3 - \{0\}$. Compute the expressions of the tangent vectors $\frac{\partial}{\partial r}|_p$, $\frac{\partial}{\partial \theta_1}|_p$ and $\frac{\partial}{\partial \theta_2}|_p$ in the basis that consists of the vectors $\frac{\partial}{\partial x^1}|_p$, $\frac{\partial}{\partial x^2}|_p$ and $\frac{\partial}{\partial x^3}|_p$.

Solution. Let

$$\Phi\colon \,]0,\infty[\, \times \,]{-\pi},\pi[\, \times \,]{-\tfrac{\pi}{2}},\tfrac{\pi}{2}[\, \to U = \mathbb{E}^3 - \{x \in \mathbb{E}^3 : x_3 \le 0\}$$

$$\Phi(r,\theta_1,\theta_2) = (r\cos\theta_1 \sin\theta_2, r\sin\theta_1 \sin\theta_2, r\cos\theta_2) \,.$$

(Such a restriction is necessary for Φ to be injective, since the coordinates (r,θ_1,θ_2) and $(r,\theta_1+\pi,-\theta_2)$ both correspond to the same points in $\mathbb{E}^3$). The inverse mapping Φ^{-1} is a coordinate mapping. By considering the charts $\phi = \Phi^{-1}$ and $\psi = \mathrm{id}$, we get

$$\begin{aligned} \frac{\partial}{\partial r}\Big|_{\Phi(r',\theta_1',\theta_2')} &= \cos\theta_1' \sin\theta_2' \frac{\partial}{\partial x^1} + \sin\theta_1' \sin\theta_2' \frac{\partial}{\partial x^2} + \cos\theta_2' \frac{\partial}{\partial x^3} \\ \frac{\partial}{\partial \theta_1}\Big|_{\Phi(r',\theta_1',\theta_2')} &= -r'\sin\theta_1' \sin\theta_2' \frac{\partial}{\partial x^1} + r'\cos\theta_1' \sin\theta_2' \frac{\partial}{\partial x^2} \\ \frac{\partial}{\partial \theta_2}\Big|_{\Phi(r',\theta_1',\theta_2')} &= r'\cos\theta_1' \cos\theta_2' \frac{\partial}{\partial x^1} + r'\sin\theta_1' \cos\theta_2' \frac{\partial}{\partial x^2} - \sin\theta_2' \frac{\partial}{\partial x^3} \quad . \end{aligned}$$

4. Let M_1 and M_2 be smooth manifolds and let $\pi_k\colon M_1 \times M_2 \to M_k$ be the projection mappings $\pi_k(p_1, p_2) = p_k$ for $k \in \{1, 2\}$. Let $p = (p_1, p_2) \in M_1 \times M_2$. Prove that the mapping $d(\pi_1)_p \times d(\pi_2)_p\colon T_p(M_1 \times M_2) \to T_{p_1}(M_1) \times T_{p_2}(M_2)$,

$$d(\pi_1)_p \times d(\pi_2)_p(v) = (d(\pi_1)_p(v), d(\pi_2)_p(v)),$$

is a linear isomorphism.[1]

Solution. The vector spaces $T_p(M_1 \times M_2)$ and $T_{p_1}(M_1) \times T_{p_2}(M_2)$ are finite-dimensional and their dimensions are equal so we know that the spaces are isomorphic. The content of the exercise is to check that the natural mapping is an isomorphism.

Let (U_i, ϕ_i) be smooth charts at $p_i \in M_i$ for $i \in \{1, 2\}$. Let $v \in T_{p_1}M_1$. By Proposition 3.11, there is a smooth path $\gamma\colon I \to M_1$ such that $\gamma(0) = p_1$ and $\dot\gamma(0) = v$. Define the path $\tilde\gamma\colon I \to M_1 \times M_2$ by

$$\tilde\gamma(t) = (\gamma(t), p_2).$$

As $\gamma = \pi_1 \circ \tilde\gamma$, we have, as in the proof of Proposition 3.12,

$$v = \dot\gamma(0) = (\pi_1 \dot\circ \tilde\gamma)(0) = (d\pi_1)_p \dot{\tilde\gamma}(0).$$

On the other hand, the mapping $\pi_2 \circ \tilde\gamma$ is constant. Thus,

$$0 = (\pi_2 \dot\circ \tilde\gamma)(0) = (d\pi_2)_p \dot{\tilde\gamma}(0),$$

which implies $d(\pi_1)_p \times d(\pi_2)_p(\dot{\tilde\gamma}(0)) = (v, 0)$. Similarly, we can show that any vector $(0, w) \in T_{p_1}(M_1) \times T_{p_2}(M_2)$ is in the range and the claim follows by linearity.

(Using the notations and the result from Exercise 2 of the previous exercise sheet, one can prove that the inverse map of $d(\pi_1)_p \times d(\pi_2)_p$ is explicitly given by the formula $f(u, v) = d(i_{p_2})_{p_1}(u) + d(i_{p_1})_{p_2}(v)$).

Let G be a smooth manifold and that G is a multiplicative group such that the mappings $\mu\colon G \times G \to G$, $\mu(g, h) = gh$ and $\iota\colon G \to G$, $\iota(g) = g^{-1}$ are smooth. Then G is a *Lie group.*

5. Let G be a Lie group and let $e \in G$ be its neutral element. Prove that[2]

$$d\mu_{(e,e)}(v, w) = v + w$$

and[3]

$$d\iota_{(e,e)}(v) = -v.$$

Solution. By Proposition 3.11, there is a smooth path $\gamma_0\colon I \to G$ such that $\dot\gamma(0) = v$. The path $\gamma\colon I \to G \times G$, $\gamma(t) = (\gamma_0(t), e)$ is smooth and $\dot\gamma(0) = (v, 0)$.[4] Note that $\mu \circ \gamma = \gamma_0$. By Proposition 3.12, we have

$$d\mu_{(e,e)}(v, 0) = (\mu \dot\circ \gamma)(0) = \dot\gamma_0(0) = v.$$

[1] Use the product of the charts of M_1 and M_2 as charts on the product manifold as usual. Consider coordinate vectors.

[2] Compute first $d\mu_{(e,e)}(v, 0)$ using velocity vectors of paths. Note that we are identifying $T_e(G \times G)$ with $T_eG \times T_eG$.

[3] Consider the mapping $g \mapsto \mu(g, \iota(g)) = e$.

[4] Here we identify $T_{(e,e)}G \times G$ with $T_eG \times T_eG$.

Similarly, we get $d\mu_{(e,e)}(0, w) = w$ and, by linearity, $d\mu_{(e,e)}(v, w) = v + w$.

Let $\nu\colon G \to G \times G$, $\nu(g) = (g, \iota(g))$. Now $d\nu_e = (d(\mathrm{id}_G)_e, d\iota_e) = (\mathrm{id}, d\iota_e)$. Note that the function $\mu \circ \nu$ is constant equal to e, which implies

$$0 = d\mu \circ \nu_e = d\mu_{(e,e)} d\nu_e(v) = v + d\iota_e(v)$$

for for all $v \in T_eG$.