Differential geometry 2023

Exercises 12

1. (1) Let

$$\omega = dx^3 \in \Omega^1(\mathbb{S}^2)\,,$$
$$\eta = x^2x^3dx^1 + x^1x^3dx^2 + x^1x^2dx^3 \in \Omega^1(\mathbb{S}^2)$$

and let $\gamma\colon [-1,1] \to \mathbb{S}^2$,

$$\gamma(t) = \left(\sqrt{1-t^2}\cos(t), \sqrt{1-t^2}\sin(t), t\right).$$

Determine the values of the integrals $\int_\gamma \omega$ and $\int_\gamma \eta$.

Solution. By the fundamental theorem for integration on curves, since x^3 is a smooth function on $\mathbb{S}^2$, we have

$$\int_\gamma \omega = \int_\gamma dx^3 = x^3(\gamma(1)) - x^3(\gamma(-1)) = 1 - (-1) = 2.$$

We notice that $\eta = d(x^1x^2x^3)$, hence we can once again use the fundamental theorem on the curve γ and obtain

$$\int_\gamma \eta = \int_\gamma d(x^1x^2x^3) = x^1x^2x^3(\gamma(1)) - x^1x^2x^3(\gamma(-1)) = 0 - 0 = 0.$$

2. Give an example of an oriented atlas of $\mathbb{S}^n$.[1]

Solution. We begin with the atlas $\{(U_+, S_+), (U_-, S_-)\}$ given by two stereographic projections: $U_\pm = \mathbb{S}^n - \{(0,\dots,0,\pm1)\}$ and

$$S_\pm(x) = \frac{(x_1,\dots,x_{n-1})}{1 \mp x_n}$$

Their inverse is given on $\mathbb{R}^{n-1}$ by

$$S_\pm^{-1}(y) = \frac{\left(2y, \pm(\|y\|^2 - 1)\right)}{1 + \|y\|^2}.$$

We compute, for $y \neq 0$,

$$S_+ \circ S_-^{-1}(y) = \frac{y}{\|y\|^2} = S_- \circ S_+^{-1}(y).$$

Their Jacobian matrix is then given by $\mathrm{Jac}(S_\pm \circ S_\mp) : y \mapsto \frac{1}{\|y\|^2} I_n - \frac{2}{\|y\|^4}(y_iy_j)_{1\le i,j\le n}$. From linear algebra we know that, for any rank 1 matrix A, we have

$$\det(I_n + A) = 1 + \mathrm{tr}(A).$$

[1] The stereographic projections from the north and south poles form a smooth atlas that consists of two charts.

Here it gives, for all $y \neq 0$,

$$\det(\mathrm{Jac}_y(S_\pm \circ S_\mp)) = \frac{1}{\|y\|^{2n}} \det(I_n - \frac{2}{\|y\|^2}(y_i y_j)_{1\le i,j\le n}) = \frac{1}{\|y\|^{2n}}\Big(1 - \frac{2}{\|y\|^2}\sum_{i=1}^n y_i^2\Big) = -\frac{1}{\|y\|^{2n}}.$$

Thus the atlas $\{(U_\pm, S_\pm)\}$ is not oriented. However, since it only consists of 2 charts, we can use the following trick: define a linear map $\phi : \mathbb{E}^n \to \mathbb{E}^n$ by $\phi(y) = (-y_1, y_2, \ldots, y_n)$. Then the atlas $\{(U_+, S_+), (U_-, \phi \circ S_-)\}$ is an oriented atlas. Indeed, for every $y \neq 0$, we have

$$\det(\mathrm{Jac}(\phi \circ S_- \circ S_+)(y)) = \det(\phi)\det(\mathrm{Jac}(S_- \circ S_+)(y)) = -\det(\mathrm{Jac}(S_- \circ S_+)(y)) > 0.$$

Remark. Instead of computing the determinant of $\mathrm{Jac}_y(S_\pm \circ S_\mp)$ for all $y \in \mathbb{E}^n - \{0\}$, it would have been sufficient to do it for only one such point y, for example $y = (1, 0, \ldots, 0)$ (giving a diagonal matrix), and then to use the connectedness of $\mathbb{E}^n - \{0\}$ to argue that this (non vanishing anywhere) determinant has constant sign.

3. (1) Prove that the mapping $-\mathrm{id}\colon \mathbb{E}^n \to \mathbb{E}^n$ preserves orientation if and only if n is even.
(2) Prove that the mapping $-\mathrm{id}\colon \mathbb{S}^n \to \mathbb{S}^n$ preserves orientation if and only if n is odd.[2]

Solution. (1) We use the trivial atlas $(\mathbb{E}^n, \mathrm{id})$. The map $\mathrm{id} \circ (-\mathrm{id}) \circ \mathrm{id}^{-1} = -\mathrm{id}$ is linear (hence equal to its Jacobian matrix) and has determinant $(-1)^n$. Thus, it preserves orientation iff n is even.
(2) We use the oriented atlas $\{(U_+, S_+), (U_-, \phi \circ S_-)\}$ from Exercise 2. Since the map $-\mathrm{id}$ is a (global but local would be sufficient) diffeomorphism, we know (continuity of determinant) that it is sufficient to check that $-\mathrm{id}$ is order preserving in only one of these two charts. We choose (U_+, S_+) and compute, for all $y \in \mathbb{E}^n$,

$$S_+ \circ (-\mathrm{id}) \circ S_+^{-1}(y) = S_+\Big(-\frac{(2y, \|y\|^2 - 1)}{1 + \|y\|^2}\Big) = -\frac{y}{\|y\|^2}.$$

Hence we have the formula for its Jacobian matrix: $\mathrm{Jac}(S_+ \circ (-\mathrm{id}) \circ S_+^{-1}) = -\mathrm{Jac}(y \mapsto \frac{y}{\|y\|^2})$. Using the computation of the Jacobian of $y \mapsto \frac{y}{\|y\|^2}$ from Exercise 2, we obtain, for all $y \in \mathbb{E}^n$,

$$\det(\mathrm{Jac}(S_+ \circ (-\mathrm{id}) \circ S_+^{-1})(y)) = (-1)^n \det(y' \mapsto \frac{y'}{\|y'\|^2}(y)) = (-1)^{n+1}.$$

The result follows.

4. Prove that the n-torus $\mathbb{T}^n = \mathbb{E}^n/\mathbb{Z}^n$ is orientable.[3]

Solution. The standard atlas on $\mathbb{T}^n$ is given by

$$\{\Big(\pi(B_\infty(x, \frac{1}{2})), (\pi_{|B_\infty(x,\frac{1}{2})})^{-1}\Big) \;:\; x \in \mathbb{E}^n\}$$

where $\pi : \mathbb{E}^n \to \mathbb{T}^n$ is the canonical projection, which is locally invertible. Let $x, y \in \mathbb{E}^n$. We see that, for all $z \in \pi^{-1}(\pi(B(x, \frac{1}{2}))) \cap B(y, \frac{1}{2})$, there exists $k \in \mathbb{Z}^n$ such that $(\pi_{|B(x,\frac{1}{2})})^{-1} \circ ((\pi_{|B(y,\frac{1}{2})})^{-1})^{-1}(z) = z + k$. Since the latter composed map is smooth, the integer k does not depend on z. Hence, the Jacobian of $(\pi_{|B(x,\frac{1}{2})})^{-1} \circ ((\pi_{|B(y,\frac{1}{2})})^{-1})^{-1}$ is constant equal to 1, and the standard atlas on $\mathbb{T}^n$ is already oriented.

[2]Use the oriented atlas from Problem 2 .
[3]Recall that local inverses of the quotient mapping $\pi\colon \mathbb{E}^n \to \mathbb{T}^n$ form a smooth atlas.

5. The mapping $G\colon \mathbb{T}^2 \to \mathbb{E}^3$ induced by the mapping $\widetilde{G}\colon \mathbb{E}^2 \to \mathbb{E}^3$,

$$\widetilde{G}(x) = \Big((2+\cos(2\pi x^1))\cos(2\pi x^2),\, (2+\cos(2\pi x^1))\sin(2\pi x^2),\, \sin(2\pi x^1)\Big)\,,$$

is a smooth embedding of the 2-torus $\mathbb{T}^2$ into $\mathbb{E}^3$. Compute

$$\int_{\mathbb{T}^2} G^*(x^3 dx^1 \wedge dx^2)\,.$$

Solution.
Solution 1.
In this solution, we find an explicit formula for the form $G^*(x^3dx^1 \wedge dx^2)$, and then we integrate it. We denote by $ds^1 \wedge ds^2$ the usual orientation form on $\mathbb{T}^2$, obtained by the charts in Exercise 4. We compute

$$\begin{aligned}
G^*(x^3dx^1 \wedge dx^2) =& x^3(G(s))\Big(\sum_{i=1}^{2} \frac{\partial \widetilde{G}_1}{\partial s^i} ds^i\Big) \wedge \Big(\sum_{j=1}^{2} \frac{\partial \widetilde{G}_2}{\partial s^j} ds^j\Big)\\
=& \sin(2\pi s^1)\Big(-2\pi\sin(2\pi s^1)\cos(2\pi s^2)ds^1 - (2+\cos(2\pi s^1))2\pi\sin(2\pi s^2)ds^2\Big)\\
&\wedge\Big(-2\pi\sin(2\pi s^1)\sin(2\pi s^2)ds^1 + (2+\cos(2\pi s^1))2\pi\cos(2\pi s^2)ds^2\Big)\\
=& -4\pi^2\sin(2\pi s^1)^2(2+\cos(2\pi s^1))\,ds^1 \wedge ds^2.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\int_{\mathbb{T}^2} G^*(x^3dx^1 \wedge dx^2) &= -4\pi^2 \int_0^1\int_0^1 \sin(2\pi s_1)^2(2+\cos(2\pi s_2))\,ds_1ds_2\\
&= -4\pi^2 \int_0^1 \sin(2\pi s_1)^2\,ds_1 \times \int_0^1 (2+\cos(2\pi s_2))\,ds_2\\
&= -4\pi^2\,\frac{1}{2} \times 2 = -4\pi^2.
\end{aligned}$$

Solution 2 (with Stokes's theorem).
The torus $G(\mathbb{T}^2)$ is the usual embedded torus in $\mathbb{E}^3$ or radii 2 and 1. Let us denote by T the associated solid torus. The Stokes orientation on the torus $G(\mathbb{T}^2)$ is given by the one on $\mathbb{E}^3$ and outward normal vectors based on $\partial T = G(\mathbb{T}^2)$. Using this orientation on $G(\mathbb{T}^2)$ and the orientation on $\mathbb{T}^2$ defined in Exercise 4 (hence G preserves orientation iff $G \circ \pi = \widetilde{G}$ does), we check the preserving/reversing of orientation of $\widetilde{G}$ at the point $x = (0,0)$, thus $G(x) = (3,0,0)$ and an associated outward pointing vector is $(1,0,0)$. We get

$$\det\Big((1,0,0),\, d\widetilde{G}_x(1,0),\, d\widetilde{G}_x(0,1)\Big) = \begin{vmatrix} 1 & 0 & 0\\ 0 & 0 & 3\pi\\ 0 & 2\pi & 0\end{vmatrix} = -6\pi^2 < 0.$$

Thus G is orientation reversing. Then, by the pullback (or "change of variable") formula for integration of forms on manifold applied to the embedding $\widetilde{G}$, we obtain

$$\int_{\mathbb{T}^2} G^*(x^3dx^1 \wedge dx^2) = -\int_{G(\mathbb{T}^2)} x^3dx^1 \wedge dx^2.$$

By Stokes's Theorem, we get

$$\int_{G(\mathbb{T}^2)} x^3dx^1 \wedge dx^2 = \int_T dx^3 \wedge dx^1 \wedge dx^2 = \int_T dx^1 \wedge dx^2 \wedge dx^3 = \int_T dx_1dx_2dx_3 = \mathrm{vol}(T).$$

where T has volume $(2\pi \times 2) \times (\pi \times 1^2) = 4\pi^2$ (to find the general formula for the volume of a torus, you may use a polar change of variable twice, first from (x_1, x_2) to (r, θ), then from (r, x_3) to (ρ, ω)). In the end, we obtain

$$\int_{\mathbb{T}^2} G^*(x^3dx^1 \wedge dx^2) = -4\pi^2.$$