Observe/recall that uncountable union of zero measurable sets is not necessarily zero measurable, cf. $m(\cup_{x\in(0,1)}\{x\}) = 1$. Therefore the restriction on the countable set of cubes was necessary above.

Example 4.9. *$w(x) = |x|^{-\alpha}$, $0 \le \alpha < n$, $x \in \mathbf{R}^n$, belongs to A_1. Indeed, let $x \in \mathbf{R}^n\setminus\{0\}$, $x \in Q$. Then by choosing a radius $r = l(Q)\sqrt{n}$, we see that*

$$Q \subset B(x, r).$$

We calculate

$$\begin{aligned}
\frac{1}{m(Q)}\int_{Q\ni x} w(y)\,\mathrm{d}y &\le \frac{C}{B(x,r)}\int_{B(x,r)} w(y)\,\mathrm{d}y \\
&\overset{z=\frac{y}{|x|},y=z|x|,\ \mathrm{d}y=|x|^n\ \mathrm{d}z}{=} \frac{C}{r^n}\int_{B(\frac{x}{|x|},\frac{r}{|x|})} ||x|\,z|^{-\alpha}\,|x|^n\ \mathrm{d}z \\
&= \frac{C\,|x|^{-\alpha}}{\left(\frac{r}{|x|}\right)^n}\int_{B(\frac{x}{|x|},\frac{r}{|x|})} |z|^{-\alpha}\ \mathrm{d}z \\
&\le Cw(x)\underbrace{Mw\Big(\frac{x}{|x|}\Big)}_{<\infty}.
\end{aligned}$$

Thus by taking a supremum over Q such that $x \in Q$, we see that

$$Mw(x) \le Cw(x),$$

so that by Lemma 4.8, $w \in A_1$.

Next we derive **a necessary condition for weak** (p,p) estimate to hold.
Lemma 4.4 gives us the estimate

$$\mu(Q)\Big(\frac{1}{m(Q)}\int_Q |f(x)|\ \mathrm{d}x\Big)^p \le C\int_Q |f(x)|^p\ \mathrm{d}\mu.$$

We choose $f(x) = w^{1-p'}(x)$, where $1/p' + 1/p = 1$ i.e. $p' = p/(p-1)$. Recalling that $\mu(Q) = \int_Q w(x)\,\mathrm{d}x$, we get

$$\begin{aligned}
\int_Q w(x)\,\mathrm{d}x\Big(\frac{1}{m(Q)}\int_Q w^{1-p'}(x)\,\mathrm{d}x\Big)^p &\le C\int_Q w^{(1-p')p}(x)w(x)\,\mathrm{d}x \\
&= C\int_Q w(x)^{(1-p')p+1}\,\mathrm{d}x.
\end{aligned}$$

A short calculation $((1-p')p + 1 = (1-p/(p-1))p + 1 = ((p-1-p)/(p-1))p + 1 = -p/(p-1) + 1 = 1 - p')$ shows that

$$(1-p')p + 1 = 1 - p'$$

so that if we divide by the integral on the right hand side the above inequality, we get

$$\frac{1}{m(Q)}\int_Q w(x)\,\mathrm{d}x\Big(\frac{1}{m(Q)}\int_Q w^{1-p'}(x)\,\mathrm{d}x\Big)^{p-1}\le C, \tag{4.10}$$

or

$$\frac{1}{m(Q)}\int_Q w(x)\,\mathrm{d}x\Big(\frac{1}{m(Q)}\int_Q w^{1/(1-p)}(x)\,\mathrm{d}x\Big)^{p-1}\le C.$$

This is called the *Muckenhoupt A_p-condition.*

Observe that above, we implicitly use $w^{1-p'}\in L^1_{\mathrm{loc}}(\mathbf{R}^n)$. If this is not the case, we can consider

$$f=(w+\varepsilon)^{1-p'},$$

derive the above estimate, and let finally $\varepsilon\to 0$. After this argument, as $w>0$ a.e., (4.10) implies that $w^{1-p'}\in L^1_{\mathrm{loc}}(\mathbf{R}^n)$.

Let us collect the above definitions.

Definition 4.11 (Muckenhoupt 1972)**.** Let $w\in L^1_{\mathrm{loc}}(\mathbf{R}^n)$, $w>0$ a.e. Then w satisfies A_1-condition if there exists $C>0$ s.t.

$$\int_Q w(x)\,\mathrm{d}x\le C\operatorname*{ess\,inf}_{y\in Q} w(y).$$

for all cubes $Q\subset\mathbf{R}^n$. For $1<p<\infty$, w satisfies A_p-condition if there exists $C>0$ s.t.

$$\frac{1}{m(Q)}\int_Q w(x)\,\mathrm{d}x\Big(\frac{1}{m(Q)}\int_Q w^{1-p'}(x)\,\mathrm{d}x\Big)^{p-1}\le C$$

for all cubes $Q\subset\mathbf{R}^n$.

Remark 4.12. (i) $1-p'=1/(1-p)<0$, $w^{1-p'}\in L^1_{\mathrm{loc}}(\mathbf{R}^n)$

(ii) Let $p=2$. Then

$$\frac{1}{m(Q)}\int_Q w(x)\,\mathrm{d}x\frac{1}{m(Q)}\int_Q \frac{1}{w(x)}\,\mathrm{d}x\le C$$

(iii)

$$\begin{aligned} m(Q)&=\int_Q w^{1/p}w^{-1/p}\,\mathrm{d}x\\ &\overset{\text{Hölder}}{\le}\Big(\int_Q w^{p(1/p)}\,\mathrm{d}x\Big)^{1/p}\Big(\int_Q w^{p'(-1/p)}\,\mathrm{d}x\Big)^{1/p'}\\ &=\Big(\int_Q w\,\mathrm{d}x\Big)^{1/p}\Big(\int_Q w^{1-p'}\,\mathrm{d}x\Big)^{1/p'}.\end{aligned}$$

Dividing by $m(Q)=m(Q)^{1/p}m(Q)^{1/p'}$ and then taking power p on both sides we get

$$\frac{1}{m(Q)}\int_Q w\,\mathrm{d}x\Big(\frac{1}{m(Q)}\int_Q w^{1-p'}\,\mathrm{d}x\Big)^{p-1}\ge 1 \tag{4.13}$$

so that

$$\Big(\frac{1}{m(Q)}\int_Q w^{1-p'}\,\mathrm{d}x\Big)^{1-p} \le \frac{1}{m(Q)}\int_Q w(x)\,\mathrm{d}x.$$

This was (a consequence of) Hölder's inequality. On the other hand, by looking at the A_p condition, we see that the inequality is reversed. Thus A_p condition is a reverse Hölder's inequality.

Theorem 4.14. *$A_p \subset A_q$, $1 \le p < q$.*

Proof. **Case $1 < p < \infty$.** We recall that $q' - 1 = 1/(q-1)$.

$$\begin{aligned}
&\Big(\frac{1}{m(Q)}\int_Q \Big(\frac{1}{w}\Big)^{\frac{1}{q-1}}\,\mathrm{d}x\Big)^{q-1}\\
&\quad\overset{\text{Hölder}}{\le} \Big(\frac{1}{m(Q)}\Big)^{q-1}\Big(\int_Q \Big(\frac{1}{w}\Big)^{\frac{1}{q-1}\frac{q-1}{p-1}}\,\mathrm{d}x\Big)^{(q-1)\frac{p-1}{q-1}} m(Q)^{(q-1)(1-\frac{p-1}{q-1})}\\
&\quad= C\Big(\int_Q \Big(\frac{1}{w}\Big)^{1/(p-1)}\,\mathrm{d}x\Big)^{p-1} m(Q)^{1-p}\\
&\quad\overset{w\,\in\,A_p}{\le} \Big(\frac{1}{m(Q)}\int_Q w\,\mathrm{d}x\Big)^{-1}
\end{aligned}$$

which proves the claim in this case.
Case $p = 1$.

$$\begin{aligned}
\Big(\frac{1}{m(Q)}\int_Q \Big(\frac{1}{w}\Big)^{1/(q-1)}\,\mathrm{d}x\Big)^{q-1} &\le \operatorname*{ess\,sup}_Q \frac{1}{w}\\
&= \frac{1}{\operatorname{ess\,inf}_Q w} \overset{w\,\in\,A_1}{\le} \frac{C}{⨍_Q w\,\mathrm{d}x}. \qquad \square
\end{aligned}$$

Theorem 4.15. *Let $1 \le p < \infty$, and $w \in L^1_{loc}(\mathbf{R}^n)$, $w < 0$ a.e. Then $w \in A_p$ if and only if*

$$\Big(\frac{1}{m(Q)}\int_Q |f(x)|\,\mathrm{d}x\Big)^p \le \frac{C}{\mu(Q)}\int_Q |f(x)|^p\,\mathrm{d}\mu.$$

for every $f \in L^1_{loc}(\mathbf{R}^n)$ and $Q \subset \mathbf{R}^n$.

Proof. **Case $1 < p < \infty$.**

"$\Leftarrow$" was already proven before (4.10).

"$\Rightarrow$" First we use Hölder's inequality

$$\begin{aligned}
\frac{1}{m(Q)}\int_Q |f(x)|\,\mathrm{d}x &= \frac{1}{m(Q)}\int_Q |f(x)|\,w(x)^{1/p}\Big(\frac{1}{w(x)}\Big)^{1/p}\,\mathrm{d}x\\
&\le \frac{1}{m(Q)}\Big(\int_Q |f(x)|^p\,w(x)\,\mathrm{d}x\Big)^{1/p}\Big(\int_Q \Big(\frac{1}{w(x)}\Big)^{p'/p}\,\mathrm{d}x\Big)^{1/p'},
\end{aligned}$$

for $1/p' + 1/p = 1$. By taking the power p on both sides, using the definition of μ, arranging terms, using $p/p' = p-1$, $-p'/p = 1/(1-p)$, and A_p condition, we get

$$\begin{aligned}\mu(Q)\Big(\frac{1}{m(Q)}\int_Q |f(x)|\,\mathrm{d}x\Big)^p &\le \frac{1}{m(Q)^p}\Big(\int_Q |f(x)|^p\,w(x)\,\mathrm{d}x\Big)\\ &\quad\cdot \underbrace{\int_Q w(x)\,\mathrm{d}x\Big(\int_Q w(x)^{1/(1-p)}\,\mathrm{d}x\Big)^{p-1}}_{\overset{w\,\in\,A_p}{\le}\,Cm(Q)^p}\\ &\le C\int_Q |f(x)|^p\,\mathrm{d}\mu.\end{aligned}$$

Case $p = 1$.

"$\Leftarrow$" was already proven before (4.7).

"$\Rightarrow$" Let $w \in A_1$ i.e.

$$\frac{1}{m(Q)}\int_Q w(x)\,\mathrm{d}x \le C \operatorname*{ess\,inf}_{x\in Q} w(x).$$

Then

$$\begin{aligned}\mu(Q)\frac{1}{m(Q)}\int_Q |f(x)|\,\mathrm{d}x &\le \frac{1}{m(Q)}\int_Q |f(x)|\,\mu(Q)\,\mathrm{d}x\\ &\overset{w\,\in\,A_1}{\le} \int_Q |f(x)| \operatorname*{ess\,inf}_{x\in Q} w(x)\,\mathrm{d}x\\ &\le C\int_Q |f(x)|\,w(x)\,\mathrm{d}x\\ &\le C\int_Q |f(x)|\,\mathrm{d}\mu. \qquad \square\end{aligned}$$

We aim at proving that the weighted weak/strong type estimate and A_p condition are equivalent. To establish this, we next study Calderón-Zygmund decomposition. It is an important tool both in harmonic analysis and in the theory of PDEs.

30.9.2010

4.1. **Calderón-Zygmund decomposition.** Next we introduce dyadic cubes, which are generated using powers of 2.

Definition 4.16 (Dyadic cubes)**.** In this section we integrate with respect to the measure m only, and thus we recall the notation $⨍_Q = \frac{1}{m(Q)}\int_Q$.

A dyadic interval on $\mathbf{R}$ is

$$[m2^{-k}, (m+1)2^{-k})$$