To establish this, we calculate

$$\begin{aligned}|x-z|^2 &\leq (|x-y|+|y-z|)^2\\ &\overset{\text{convexity}}{\leq} 2(|x-y|^2+|y-z|^2)\\ &\leq 2((\alpha t)^2+|y-z|^2).\end{aligned}$$

Thus

$$\begin{aligned}|x-z|^2+t^2 &\leq (2\alpha^2+1)t^2+2\,|y-z|^2\\ &\leq \max(2,2\alpha^2+1)(|y-z|^2+t^2)\end{aligned}$$

so that

$$\frac{|x-z|^2+t^2}{\max(2,2\alpha^2+1)} \leq (|y-z|^2+t^2).$$

We apply this and deduce

$$\begin{aligned}P_t(y-z) &= C(n)\frac{t}{(|y-z|^2+t^2)^{(n+1)/2}}\\ &\leq C(n)\max(2,2\alpha^2+1)^{(n+1)/2}\frac{t}{(|x-z|^2+t^2)^{(n+1)/2}}\\ &= C(n,\alpha)P_t(x-z).\end{aligned}$$

Utilizing this result we attack the original question and estimate

$$\begin{aligned}|u(y,t)| &\leq \int_{\mathbf{R}^n} |f(z)|\, P_t(y-z)\,\mathrm{d}z\\ &\leq C(\alpha,n)\int_{\mathbf{R}^n} |f(z)|\, P_t(x-z)\,\mathrm{d}z\\ &= C(\alpha,n)(|f| * P_t)(x)\\ &\leq C(\alpha,n)\sup_{t>0}(|f| * P_t)(x)\\ &\overset{\text{Theorem 3.10}}{\leq} C(\alpha,n)Mf(x).\end{aligned}$$

This concludes the proof giving

$$\sup_{(x,t)\in\Gamma_\alpha(x)} |u(y,t)| \leq cMf(x).$$

□

Corollary 3.21. *If* $f \in L^p(\mathbf{R}^n)$*,* $1 \leq p \leq \infty$*, then*

$$(f * P_t)(y) \to f(x)$$

nontangentially for almost every $x \in \mathbf{R}^n$*.*

Proof. Replace in (3.16) the use of Theorem 3.10 by the above estimate.

□

Remark 3.22. By considering a discontinuous $f \in L^p$, we see that $(f * P_{t_n})(y_n)$ does not converge to $f(x)$ for every sequence $(y_n, t_n) \to (x, 0)$. The cone is not the whole of the half space i.e. α must be finite!

Nevertheless, if $f \in C(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$, it follows that

$$u(y,t) = (f * P_t)(y) \to f(x)$$

when $(y,t) \to (x,0)$ in $\mathbf{R}^{n+1}_+$ without further restrictions. This is a consequence of Remark 3.9.

4. Muckenhoupt weights

A weight is a function $w \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$, such that $w \geq 0$ a.e. We have already seen that strong (p,p) property for a Hardy-Littlewood maximal function is an important tool in many applications. Next we study the question in the weighted case:

Let $1 < p < \infty$. Which weights $w \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ satisfy

$$\int_{\mathbf{R}^n} (Mf(x))^p w(x)\,\mathrm{d}x \leq C \int_{\mathbf{R}^n} |f(x)|^p\, w(x)\,\mathrm{d}x? \tag{4.1}$$

for every $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$. As before

$$Mf(x) = \sup_{Q \ni x} \frac{1}{m(Q)} \int_Q |f(y)|\ \mathrm{d}y$$

is a Hardy-Littlewood maximal function.

This estimate implies the weak (p,p) estimate. Indeed,

$$\begin{aligned}
\int_{\{x \in \mathbf{R}^n \,:\, Mf(x) > \lambda\}} w(x)\,\mathrm{d}x &\leq \int_{\{x \in \mathbf{R}^n \,:\, Mf(x) > \lambda\}} \Big(\frac{Mf(x)}{\lambda}\Big)^p w(x)\,\mathrm{d}x \\
&\leq \frac{1}{\lambda^p} \int_{\mathbf{R}^n} (Mf(x))^p w(x)\,\mathrm{d}x \qquad (4.2)\\
&\overset{(4.1)}{\leq} \frac{C}{\lambda^p} \int_{\mathbf{R}^n} |f(x)|^p\, w(x)\,\mathrm{d}x.
\end{aligned}$$

If we define a measure

$$\mu(E) := \int_E w(x)\,\mathrm{d}x$$

then the weighted strong (p,p) estimate (4.1) can be written as

$$\int_{\mathbf{R}^n} (Mf(x))^p\,\mathrm{d}\mu \leq C \int_{\mathbf{R}^n} |f(x)|^p\ \mathrm{d}\mu \tag{4.3}$$

First, we derive some consequences for the weighted **weak** (p,p) estimate. Thus we also obtain some necessary conditions for the question: Which weights $w \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ satisfy weak (p,p) type estimate?

Lemma 4.4. *Suppose that the weighted weak* (p, p) *estimate* (4.2) *holds for some* p, $1 \le p < \infty$. *Then*

$$\Big(\frac{1}{m(Q)}\int_Q |f(x)|\,\mathrm{d}x\Big)^p \le \frac{C}{\mu(Q)}\int_Q |f(x)|^p\,\mathrm{d}\mu$$

for all cubes $Q \subset \mathbf{R}^n$ *and* $f \in L^1_{loc}(\mathbf{R}^n)$.

Proof. Fix a cube. If $\int_Q |f(x)|\,\mathrm{d}x = 0$ or $\int_Q |f(x)|\,\mathrm{d}\mu(x) = \infty$ then the result immediately follows. Thus we may assume

$$\frac{1}{m(Q)}\int_Q |f(x)|\,\mathrm{d}x > \lambda > 0$$

which implies according to the definition of the maximal function that

$$Mf(x) > \lambda > 0$$

for every $x \in Q$. In other words,

$$Q \subset \{x \in \mathbf{R}^n \,:\, Mf(x) > \lambda\}$$

so that

$$\begin{aligned}\mu(Q) &\le \mu(\{x \in \mathbf{R}^n \,:\, Mf(x) > \lambda\})\\ &\overset{(4.2)}{\le} \frac{C}{\lambda^p}\int_{\mathbf{R}^n} |f(x)|^p\,\mathrm{d}\mu.\end{aligned}$$

If we replace f by $f\chi_Q$ then this gives

$$\mu(Q) \le \frac{C}{\lambda^p}\int_Q |f(x)|^p\,\mathrm{d}\mu,$$

and by recalling the definition of λ we get the claim. □

Remark 4.5. By analyzing the previous result, we see some of the properties of weights we are studying. Let us choose $f = \chi_E$, $E \subset Q$ a measurable set, in the previous lemma. Then the lemma gives

$$\mu(Q)\Big(\frac{m(E)}{m(Q)}\Big)^p \le C\mu(E). \tag{4.6}$$

This implies

(i) Either $w = 0$ a.e. or $w > 0$ a.e. in Q

Indeed, otherwise it would hold for

$$E = \{x \in Q \,:\, w(x) = 0\}$$

that

$$m(E), m(Q \setminus E) > 0$$

(if "$w = 0$ a.e. in Q" is false, then $m(Q \setminus E) > 0$ and similarly for the other case) and further by $m(Q \setminus E) > 0$ it follows that

$$\mu(Q) > 0.$$

Then the right hand side would be zero (clearly $\mu(E) = \int_E w(x)\,\mathrm{d}x = \int_{\{w=0\}} w\,\mathrm{d}x = 0$) whereas the left hand side would be positive. A contradiction.

(ii) By choosing $Q = Q(x, 2l)$ and $E = Q(x, l)$, we see that

$$\mu(Q(x, 2l)) \le C\mu(Q(x, l)),$$

because $m(Q(x, l))/m(Q(x, 2l)) = 2^n$. Measures with this property are called *doubling measures.*

(iii) Either $w = \infty$ a.e. or $w \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$.

If there would be a set

$$E \subset Q \text{ such that } w(x) < \infty \text{ and } m(E) > 0,$$

by (4.6) it follows that $\mu(Q) = \int_Q w(x)\,\mathrm{d}x$ is finite, and thus

$$w \in L^1(Q)$$

and by choosing larger cubes, we get $w \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$. Thus the result follows.

Observe that $w \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ was one of our assumptions when defining weights, but it would be possible to take the weak type estimate as a starting point and then derive this as a result as shown above.

Next we derive **a necessary condition for weak** $(1, 1)$ estimate to hold.

Case $p = 1$: We shall use notation

$$\operatorname*{ess\,inf}_{x \in Q} w(x) := \sup\{m \in \mathbf{R} \,:\, w(x) \ge m \text{ a.e. } x \in Q\}$$

and define a set

$$E_\varepsilon = \{x \in Q \,:\, w(x) < \operatorname*{ess\,inf}_{y \in Q} w(y) + \varepsilon\}$$

for some $\varepsilon > 0$. By definition of ess inf, we have $m(E_\varepsilon) > 0$.

Now by (4.6),

$$\begin{aligned} \frac{\mu(Q)}{m(Q)} &\le C \frac{\mu(E_\varepsilon)}{m(E_\varepsilon)} \\ &\overset{\text{def of } \mu}{=} \frac{C}{m(E_\varepsilon)} \int_{E_\varepsilon} w(x)\,\mathrm{d}x \le C(\operatorname*{ess\,inf}_{y \in Q} w(y) + \varepsilon). \end{aligned}$$

By passing to a zero with ε, and recalling that $\mu(Q) = \int_Q w(x)\,\mathrm{d}x$, we get *Muckenhoupt A_1-condition*

$$\frac{1}{m(Q)} \int_Q w(x)\,\mathrm{d}x \le C \operatorname*{ess\,inf}_{y \in Q} w(y). \tag{4.7}$$

If this condition holds we denote $w \in A_1$.

Lemma 4.8. *A weight w satisfies Muckenhoupt A_1-condition if and only if*

$$Mw(x) \le Cw(x)$$

for almost every $x \in \mathbf{R}^n$.

On the other hand from the Lebesgue density theorem, we get $w(x) \le Mw(x)$ for almost every $x \in \mathbf{R}^n$ so that

$$w(x) \le Mw(x) \le Cw(x).$$

Proof. "$\Leftarrow$" Suppose that $Mw(x) \le Cw(x)$ for almost every $x \in \mathbf{R}^n$. Then

$$\frac{1}{m(Q)} \int_Q w(y)\,\mathrm{d}y \le Cw(x) \text{ a.e. } x \in Q,$$

and thus

$$\frac{1}{m(Q)} \int_Q w(y)\,\mathrm{d}y \le C \operatorname*{ess\,inf}_{x \in Q} w(x).$$

28.9.2010 "$\Rightarrow$" Suppose that $w \in A_1$ so that $\frac{1}{m(Q)} \int_Q w(y)\,\mathrm{d}y \le C \operatorname{ess\,inf}_{x \in Q} w(x)$. We shall show that

$$m(\{x \in \mathbf{R}^n : Mw(x) > Cw(x)\}) = 0.$$

Choose a point $x \in \{x \in \mathbf{R}^n : Mw(x) > Cw(x)\}$ so that $Mw(x) > Cw(x)$. Then there exists a cube $Q \ni x$ such that

$$\frac{1}{m(Q)} \int_Q w(y)\,\mathrm{d}y > Cw(x).$$

Without loss of generality we may choose this cube so that the corners lie in the rational points. Thus

$$Cw(x) < \frac{1}{m(Q)} \int_Q w(y)\,\mathrm{d}y \overset{A_1}{\le} C \operatorname*{ess\,inf}_{y \in Q} w(y)$$

so that

$$w(x) < \operatorname*{ess\,inf}_{y \in Q} w(y).$$

For this cube, we denote by

$$E_Q = \{x \in Q : w(x) < \operatorname*{ess\,inf}_{y \in Q} w(y)\})$$

which is of measure zero. Now we repeat the process for each $x \in \{x \in \mathbf{R}^n : Mw(x) > Cw(x)\}$ and as we restricted ourselves to a countable family of cubes with corners at rational points, we have

$$m(\bigcup E_Q) = 0$$

because countable union of zero measurable sets has a measure zero. □