HARMONIC ANALYSIS, 6. EXERCISE

1. (i) Find a Fourier transform for $g(x) = xe^{-\pi x^2}$.
   (ii) Find a Fourier transform for $g(x) = x^2 e^{-\pi x^2}$.
2. Show that if $f \in L^1(\mathbf{R})$, then $\hat{f}$ is uniformly continuous. Hint: It holds (why?)

$$\left|e^{\theta i} - 1\right| \leq \begin{cases} |\theta|, & |\theta| < R, \\ 2, & \text{otherwise.} \end{cases}$$

3. Let us consider the Fourier transform in $\mathbf{R}^n$ i.e.

$$\int_{\mathbf{R}^n} f(x) e^{-2\pi i \mathbf{x}\cdot\xi}\,\mathrm{d}x.$$

   Find out how $S(\mathbf{R}^n)$ is defined. Show for $f \in S(\mathbf{R}^n)$ that

$$\widehat{\left(\frac{\partial f}{\partial x_j}\right)}(\xi) = 2\pi i \xi_j \hat{f}(\xi).$$

4. Let us consider the Fourier transform and convolution $\int_{\mathbf{R}^n} f(x-y)g(y)\,\mathrm{d}y$ in $\mathbf{R}^n$. Show that (and give suitable assumptions on the functions)

$$\widehat{f * g} = \hat{f}\hat{g}.$$

5. Consider

$$\begin{cases} \Delta u(x,t) = 0, & (x,t) \in \mathbf{R}^{n+1}_+ \\ u(x,0) = f(x), & x \in \partial \mathbf{R}^{n+1}_+ = \mathbf{R}^n. \end{cases}$$

   Derive formally by using the Fourier transform wrt. $x \in \mathbf{R}^n$ the equation

$$\begin{cases} \frac{\partial^2 \hat{u}}{\partial t^2} - 4\pi^2 |\xi|^2 \hat{u} = 0 \\ \hat{u}(\xi, 0) = \hat{f}(\xi). \end{cases}$$

   The solution to this ordinary differential equation is

$$\hat{u}(\xi,t) = \hat{f}(\xi)\hat{P}(\xi,t),$$

   where $\hat{P}(\xi,t) = e^{-2\pi|\xi|t}$. It then holds that

$$P(x,t) = C(n)\frac{t}{(|x|^2 + t^2)^{(n+1)/2}}.$$

   What is (again formally) u?

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6. (Bonus) At the first lecture, we wanted to recover a signal f_1 with a certain frequency that was mixed with other signals (we only knew the mixed signal f). This was done by multiplying the Fourier transform by a filter i.e. $a_1\hat{f}(\xi)$. We stated that formally f_1 is given by
$$c\int_{\mathbf{R}} \frac{\sin(Cy)}{y} f(x-y)\,\mathrm{d}y.$$
Let $a_1 = \chi_{(-\frac{1}{2},\frac{1}{2})}(x)$ and derive formally the above formula. What are the values c and C? Give a suitable assumption for f and justify your steps.
7. (Bonus) Write down the details (skipped during the lectures) to the fact that if $f \in L^2(\mathbf{R})$ then
$$f(x) = \lim_{R\to\infty} \int_{\{|\xi|<R\}} \hat{f}(\xi) e^{2\pi i x\xi}\,\mathrm{d}\xi.$$