Thus

$$\begin{aligned}
&\int_{\mathbf{R}^n} |(f*\phi_\varepsilon)(x) - af(x)|^p \,\mathrm{d}x \\
&\quad = \int_{\mathbf{R}^n} \left| \int_{\mathbf{R}^n} (f(x-y) - f(x))\phi_\varepsilon(y)\,\mathrm{d}y \right|^p \mathrm{d}x \\
&\quad \le \int_{\mathbf{R}^n} \Big(\int_{\mathbf{R}^n} |f(x-y) - f(x)|\, |\phi_\varepsilon(y)|^{1/p} \, |\phi_\varepsilon(y)|^{1/p'} \,\mathrm{d}y \Big)^p \mathrm{d}x \\
&\quad \overset{\text{Hölder}}{\le} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} |(f(x-y) - f(x))|^p \, |\phi_\varepsilon(y)| \,\mathrm{d}y \Big(\int_{\mathbf{R}^n} |\phi_\varepsilon(y)| \,\mathrm{d}y \Big)^{p/p'} \mathrm{d}x \\
&\quad \overset{\text{Fubini}}{=} ||\phi||_1^{p/p'} \int_{\mathbf{R}^n} |\phi_\varepsilon(y)| \Big(\int_{\mathbf{R}^n} |f(x-y) - f(x)|^p \,\mathrm{d}x \Big) \mathrm{d}y.
\end{aligned}$$

This confirms (3.8), and we start estimating I_2 and I_1.

Fix $\eta > 0$. First we estimate I_1. By a well-known result in L^p-theory, $C_0(\mathbf{R}^n)$ (compactly supported continuous functions) are dense in $L^p(\mathbf{R}^n)$ meaning that we can choose $g \in C_0(\mathbf{R}^n)$ such that

$$\int_{\mathbf{R}^n} |f(x) - g(x)|^p \,\mathrm{d}x < \eta.$$

Moreover, as g is uniformly continuous because it is compactly supported, so that we can choose small enough $r > 0$ to have

$$\int_{\mathbf{R}^n} |g(x-y) - g(x)|^p \,\mathrm{d}x < \eta,$$

for any $y \in B(0,r)$. Also recall that by convexity of $x^p, p > 1$ for some $a, b \in \mathbf{R}$ we have $|a+b|^p \le (|a| + |b|)^p = (\frac{1}{2}2\,|a| + \frac{1}{2}2\,|b|)^p \le \frac{1}{2}(2\,|a|)^p + \frac{1}{2}(2\,|b|)^p = 2^{p-1}\,|a|^p + 2^{p-1}\,|b|^p$. By using these tools, and by adding and subtracting g, we can estimate

$$\begin{aligned}
&\int_{\mathbf{R}^n} |f(x-y) - f(x)|^p \,\mathrm{d}x \\
&\quad \le \int_{\mathbf{R}^n} |f(x-y) - g(x-y) + g(x-y) - g(x) + g(x) - f(x)|^p \,\mathrm{d}x \\
&\quad \overset{\text{convexity}}{\le} C \int_{\mathbf{R}^n} |f(x-y) - g(x-y)|^p \,\mathrm{d}x \\
&\qquad + C \int_{\mathbf{R}^n} |g(x-y) - g(x)|^p \,\mathrm{d}x + C \int_{\mathbf{R}^n} |g(x) - f(x)|^p \,\mathrm{d}x \le 3\eta
\end{aligned}$$

for any $y \in B(0,r)$. Thus

$$\begin{aligned}
I_1 &= ||\phi||_1^{p/p'} \int_{B(0,r)} |\phi_\varepsilon(y)| \Big(\int_{\mathbf{R}^n} |f(x-y) - f(x)|^p \,\mathrm{d}x \Big) \mathrm{d}y \\
&\le ||\phi||_1^{p/p'} \int_{B(0,r)} |\phi_\varepsilon(y)|\, 3\eta \,\mathrm{d}y \le C\eta.
\end{aligned}$$

Next we estimate I_2. By the previous lemma (Lemma 3.6 (ii)), for any $r > 0$, there exists $\varepsilon' > 0$ such that

$$\int_{\mathbf{R}^n \setminus B(0,r)} |\phi_\varepsilon(y)| \, \mathrm{d}y < \eta,$$

for every $0 < \varepsilon < \varepsilon'$. Thus since

$$\begin{aligned}\int_{\mathbf{R}^n} |f(x-y) - f(x)|^p \, \mathrm{d}x \leq & 2^{p-1} \int_{\mathbf{R}^n} |f(x-y)|^p \, \mathrm{d}x \\ & + 2^{p-1} \int_{\mathbf{R}^n} |f(x)|^p \, \mathrm{d}x < \infty\end{aligned}$$

for $f \in L^p$, we see that

$$\begin{aligned}I_2 &= ||\phi||_1^{p/p'} \int_{\mathbf{R}^n \setminus B(0,r)} |\phi_\varepsilon(y)| \Big(\int_{\mathbf{R}^n} |f(x-y) - f(x)|^p \, \mathrm{d}x \Big) \, \mathrm{d}y \\ &\leq C \int_{\mathbf{R}^n \setminus B(0,r)} |\phi_\varepsilon(y)| \, \mathrm{d}y < C\eta,\end{aligned}$$

where $C = ||\phi||_1^{p/p'} \, 2^p \, ||f||_p^p$. Thus for any $\eta > 0$ we get an estimate

$$\int_{\mathbf{R}^n} |(f * \phi_\varepsilon)(x) - af(x)|^p \, \mathrm{d}x \leq I_1 + I_2 \leq C\eta$$

with C independent of η, by first choosing small enough r so that I_1 is small, and then for this fixed $r > 0$ by choosing ε small enough so that I_2 is small. □

Remark 3.9. Similarly, we can prove that for $\phi \in L^1(\mathbf{R}^n)$ and $a = \int_{\mathbf{R}^n} \phi \, \mathrm{d}x$, we have

(i) If $f \in C(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$, then

$$f * \phi_\varepsilon \to af$$

as $\varepsilon \to 0$ uniformly on compact subsets of $\mathbf{R}^n$.

(ii) If $f \in L^\infty(\mathbf{R}^n)$ is in addition uniformly continuous, then $f * \phi_\varepsilon$ converges uniformly to af in the whole of $\mathbf{R}^n$, that is,

$$||f * \phi_\varepsilon - af||_\infty \to 0$$

as $\varepsilon \to 0$.

Theorem 3.10. *Let $\phi \in L^1(\mathbf{R}^n)$ be such that*

(i) $\phi(x) \geq 0$ a.e. $x \in \mathbf{R}^n$.

(ii) ϕ is radial, i.e. $\phi(x) = \tilde{\phi}(|x|)$

(iii) ϕ is radially decreasing, i.e.,

$$|x| > |y| \quad \Rightarrow \quad \phi(x) \leq \phi(y).$$

Then there exists $C = C(n, \phi)$ such that

$$\sup_{\varepsilon} |((f * \phi_\varepsilon)(x)| \le CMf(x)$$

for all $x \in \mathbf{R}^n$ and $f \in L^p$, $1 \le p \le \infty$.

Proof. First we will show by a direct computation utilizing the definition of convolution, that this holds for radial functions with relatively simple structure. Then we obtain the general case by approximation argument. To this end, let us first assume that ϕ is a radial function of the form

$$\phi(x) = \sum_{i=1}^{k} a_i \chi_{B(0,r_i)}, \quad a_i > 0.$$

Then

$$\int_{\mathbf{R}^n} \phi(x)\,\mathrm{d}x = \sum_{i=1}^{k} a_i m(B(0, r_i))$$

Thus we can calculate

$$\begin{aligned}
|(f * \phi_\varepsilon)(x)| &= \left| \int_{\mathbf{R}^n} f(x-y)\phi_\varepsilon(y)\,\mathrm{d}y \right| \\
&= \left| \frac{1}{\varepsilon^n} \int_{\mathbf{R}^n} f(x-y)\phi(\frac{y}{\varepsilon})\,\mathrm{d}y \right| \\
&\overset{z=y/\varepsilon,\ \mathrm{d}y=\varepsilon^n\,\mathrm{d}z}{=} \left| \int_{\mathbf{R}^n} f(x-\varepsilon z)\phi(z)\,\mathrm{d}z \right| \\
&= \left| \sum_{i=1}^{k} \int_{B(0,r_i)} f(x-\varepsilon z) a_i\,\mathrm{d}z \right| \\
&\le \sum_{i=1}^{k} a_i \int_{B(0,r_i)} |f(x-\varepsilon z)|\ \mathrm{d}z \\
&= \sum_{i=1}^{k} a_i m(B(0,r_i)) \fint_{B(0,r_i)} |f(x-\varepsilon z)|\ \mathrm{d}z.
\end{aligned}$$

By a change of variables $y = x - \varepsilon z,\ z = (x-y)/\varepsilon,\ \mathrm{d}z = \mathrm{d}y/\varepsilon^n$ we see that

$$\begin{aligned}
\fint_{B(0,r_i)} |f(x-\varepsilon z)|\ \mathrm{d}z &= \frac{1}{\varepsilon^n m(B(0,r_i))} \int_{B(x,\varepsilon r_i)} |f(y)|\ \mathrm{d}y \\
&= \frac{1}{m(B(0,\varepsilon r_i))} \int_{B(x,\varepsilon r_i)} |f(y)|\ \mathrm{d}y \\
&\le \frac{m(Q(x,2\varepsilon r_i))}{m(B(0,\varepsilon r_i))} \frac{1}{m(Q(x,2\varepsilon r_i))} \int_{Q(x,2\varepsilon r_i)} |f(y)|\ \mathrm{d}y \\
&\le C(n)Mf(x).
\end{aligned}$$

Combining the facts, we get

$$\begin{aligned}|(f * \phi_\varepsilon)(x)| &\le \sum_{i=1}^{k} a_i m(B(0,r_i)) C(n) Mf(x) \\ &= C(n) \, ||\phi||_1 \, Mf(x).\end{aligned}$$

Next we go to the general case. As ϕ is nonnegative, radial, and radially decreasing, there exists a sequence $\phi_j, j = 1, 2, \dots$ of function as above such that $\phi_1 \le \phi_2 \le \dots$ and

$$\phi_j(x) \to \phi(x) \quad \text{a.e.} \quad x \in \mathbf{R}^n,$$

as $j \to \infty$. Now

$$\begin{aligned}|(f * \phi_\varepsilon)(x)| &\le \int_{\mathbf{R}^n} |f(x-y)| \, \phi_\varepsilon(x) \, \mathrm{d}x \\ &= \int_{\mathbf{R}^n} |f(x-y)| \lim_{j\to\infty} (\phi_j)_\varepsilon(y) \, \mathrm{d}y \\ &\overset{\text{MON}}{=} \lim_{j\to\infty} \int_{\mathbf{R}^n} |f(x-y)| \, (\phi_j)_\varepsilon(y) \, \mathrm{d}y \\ &\le C(n) \lim_{j\to\infty} ||\phi_j||_1 \, Mf(x) \\ &\overset{\text{MON}}{=} C(n) \, ||\phi||_1 \, Mf(x)\end{aligned}$$

for every $x \in \mathbf{R}^n$. In the calculation above, MON stands for the Lebesgue monotone convergence theorem. □

Remark 3.11. If ϕ is not radial or nonnegative, then we can use radial majorant

$$\tilde{\phi}(x) = \sup_{|y| \ge |x|} |\phi(y)|$$

which is nonnegative, radial and radially decreasing. Thus if $\tilde{\phi} \in L^1(\mathbf{R}^n)$, then the previous theorem, as well as the next theorem holds.

Theorem 3.12. *Let $\phi \in L^1(\mathbf{R}^n)$ be as in Theorem 3.10 that is*

(i) $\phi(x) \ge 0$ a.e. $x \in \mathbf{R}^n$.
(ii) ϕ is radial, i.e. $\phi(x) = \tilde{\phi}(|x|)$
(iii) ϕ is radially decreasing, i.e.,

$$|x| > |y| \quad \Rightarrow \quad \phi(x) \le \phi(y).$$

and $a = ||\phi||_1$. If $f \in L^p(\mathbf{R}^n)$, $1 \le p \le \infty$, then

$$\lim_{\varepsilon \to 0} (f * \phi_\varepsilon)(x) = a f(x)$$

for almost all $x \in \mathbf{R}^n$.

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