HARMONIC ANALYSIS, EXERCISE 2

1. Show that if $f \in L^p(\mathbf{R}^n)$, then $Mf(x) < \infty$ for almost all $x \in \mathbf{R}^n$. Consider all the values $1 \le p \le \infty$.
2. Show that if there is at least one point $x_0 \in \mathbf{R}^n$ such that $Mf(x_0) < \infty$, then $Mf(x) < \infty$ for almost all $x \in \mathbf{R}^n$.
3. (Bonus[1]) Let us define maximal function using balls i.e. by
$$Mf(x) := \sup_B \frac{1}{|B|} \int_{B \ni x} |f(x)| \,\mathrm{d}x.$$
Let $f : \mathbf{R}^2 \to \mathbf{R}$, $f = \chi_{B(0,1)}$ ($B(0,1)$ stands for the unit ball at the origin). Find an upper bound for $Mf(x)$ such that it is in weak $L^1(\mathbf{R}^2)$.
4. Define the maximal function using rectangles without further restriction (sides can be of any length and sides are not necessarily parallel to the coordinate axis). Show that this maximal function is not of weak type (1,1). Hint: $\mathbf{R}^2$.
5. Show that if $f \in L^1(\mathbf{R}^n)$, then
$$m(\{x \in \mathbf{R}^n \,:\, Mf(x) > \lambda\}) \le \frac{2 \cdot 5^n}{\lambda} \int_{\{x \in \mathbf{R}^n \,:\, |f(x)| > \lambda/2\}} |f(x)| \,\mathrm{d}x, \ \lambda > 0.$$
Hint: Let $g = f\chi_{\{|f| > \lambda/2\}}$. Show that $Mf(x) > \lambda \Rightarrow Mg(x) > \lambda/2$. Use Hardy-Littlewood for g.
6. Prove by estimating with simple functions that
$$\int_E |f|^p \,\mathrm{d}x = p \int_0^\infty \lambda^{p-1} m(\{x \in E \,:\, |f(x)| > \lambda\}) \,\mathrm{d}x.$$
(In the lecture, this was proven by using Fubini's theorem. Hint: Lebesgue's monotone convergence theorem, the definition of the integral and properties of measures.)

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[1]Bonus exercises are not required, but one can earn an extra point.