HARMONIC ANALYSIS, EXERCISE 1

1. State the Lebesgue's monotone convergence theorem. Show by an example that theorem fails if positivity assumption on functions in the theorem is dropped.
2. Calculate a maximal function for $f : \mathbf{R} \to \mathbf{R}$, $f(x) = \chi_{[a,b]}(x)$, $a < b$. Sketch a picture of a maximal function for
$$f(x) = \begin{cases} 1, & -2 \leq x \leq -1, \\ 1, & 1 \leq x \leq 2, \\ 0, & \text{otherwise}, \end{cases}$$
3. Show that if $Mf \in L^1(\mathbf{R}^n)$, then $f(x) = 0$ a.e. $x \in \mathbf{R}^n$.
4. Show for any increasing $\varphi \in C^1(\mathbf{R})$, $\varphi(0) = 0$ that
$$\int_{\mathbf{R}^n} \varphi(|f|)\,\mathrm{d}x = \int_0^\infty \varphi'(\lambda) m(\{x \in \mathbf{R}^n : |f(x)| > \lambda\})\,\mathrm{d}\lambda.$$
Why do we assume increasing?
5. Prove that if there is at least one point $x_0 \in \mathbf{R}^n$ such that $Mf(x_0) = 0$, then $Mf(x) = 0$ for all $x \in \mathbf{R}^n$. Give an example showing that the maximal function is nonlinear.

Date: 14.9.2010, Deadline 23.9. lecture.