(i)

$$\{x \in \mathbf{R}^n : Mf(x) > 4^n\lambda\} \subset \cup_{j=1}^{\infty} 3Q_j.$$

and

(ii)

$$\cup_{j=1}^{\infty} Q_j \subset \{x \in \mathbf{R}^n : Mf(x) > \lambda\}.$$

Proof. (i) The previous lemma.

(ii) $Q_j \in F_\lambda$ implies

$$\fint_{Q_j} |f(y)| \, \mathrm{d}y > \lambda$$

and thus

$$Mf(x) > \lambda$$

for every $x \in Q_j$. Thus

$$\cup_{j=1}^{\infty} Q_j \subset \{x \in \mathbf{R}^n : Mf(x) > \lambda\}. \qquad \square$$

4.2. **Connection of A_p to weak and strong type estimates.** Now, we return to A_p-weights.

Theorem 4.27. *Let $w \in L^1_{loc}(\mathbf{R}^n)$, and $1 \le p < \infty$. Then the following are equivalent*

(i) $w \in A_p$.

(ii)

$$\mu(\{x \in \mathbf{R}^n : Mf(x) > \lambda\}) \le \frac{C}{\lambda^p} \int_{\mathbf{R}^n} |f(x)|^p \, \mathrm{d}\mu$$

for every $f \in L^1_{loc}(\mathbf{R}^n)$, $\lambda > 0$.

Proof. It was shown above (4.10) in case $1 < p < \infty$ and in the case $p = 1$ above (4.7), that $(ii) \Rightarrow (i)$.

Then we aim at showing that $(i) \Rightarrow (ii)$. The idea is to use Lemma 4.25 and to estimate

$$\mu(\{x \in \mathbf{R}^n : Mf(x) > 4^n\lambda\}) \le \sum_{j=1}^{\infty} \mu(3Q_j), \tag{4.28}$$

for Calderón-Zygmund cubes at the level λ and for $f \in L^1(\mathbf{R}^n)$. Further, we have shown that $w \in A_p$ implies that μ is a doubling measure. Thus

$$\begin{aligned} \mu(3Q_j) &\le \mu(Q_j) \\ &\overset{\text{Theorem 4.16}}{\le} C\Big(\fint_{Q_j} |f(x)| \, \mathrm{d}x\Big)^{-p} \int_{Q_j} |f(x)|^p \, \mathrm{d}\mu(x) \\ &\overset{Q_j \text{ is a Calderón-Zygmund cube}}{\le} \frac{C}{\lambda^p} \int_{Q_j} |f(x)|^p \, \mathrm{d}\mu(x). \end{aligned}$$

Using this in (4.28), we get

$$\begin{aligned}\mu(\{x\in\mathbf{R}^n : Mf(x)>4^n\lambda\}) &\le \sum_{j=1}^{\infty}\mu(3Q_j)\\ &\le \frac{C}{\lambda^p}\sum_{j=1}^{\infty}\int_{Q_j}|f(x)|^p\,\mathrm{d}\mu(x)\\ &\overset{Q_j\text{ are disjoint}}{\le} \frac{C}{\lambda^p}\int_{\mathbf{R}^n}|f(x)|^p\,\mathrm{d}\mu(x),\end{aligned}$$

and then replacing $4^n\lambda$ by λ gives the result.

However, in the statement, we only assumed that $f\in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and in the above argument that $f\in L^1(\mathbf{R}^n)$. We treat this difficulty by considering

$$f_i = f\chi_{B(0,i)}, i=1,2,\ldots,$$

and then passing to a limit $i\to\infty$ with the help of Lebesgue's monotone convergence theorem. To be more precise, repeating the above argument, we get

$$\mu(\{x\in\mathbf{R}^n : Mf_i(x)>4^n\lambda\})\le\frac{C}{\lambda^p}\int_{\mathbf{R}^n}|f_i(x)|^p\,\mathrm{d}\mu(x).$$

Since

$$\{x\in\mathbf{R}^n : Mf(x)>4^n\lambda\}=\cup_{i=1}^{\infty}\{x\in\mathbf{R}^n : Mf_i(x)>4^n\lambda\}$$

the basic properties of measure and the above estimate imply

$$\begin{aligned}\mu(\{x\in\mathbf{R}^n : Mf(x)>4^n\lambda\}) &= \lim_{i\to\infty}\mu(\{x\in\mathbf{R}^n : Mf_i(x)>4^n\lambda\})\\ &\le \lim_{i\to\infty}\frac{C}{\lambda^p}\int_{\mathbf{R}^n}|f_i(x)|^p\,\mathrm{d}\mu\\ &\overset{\text{MON}}{=} \frac{C}{\lambda^p}\int_{\mathbf{R}^n}|f(x)|^p\,\mathrm{d}\mu. \qquad\square\end{aligned}$$

Next we show that $w\in A_p$ satisfies a reverse Hölder's inequality. First, by the usual Hölder's inequality, we get

$$\begin{aligned}\frac{1}{m(Q)}\int_Q|f(x)|\,\mathrm{d}x &\le \frac{1}{m(Q)}\Big(\int_Q|f(x)|^p\,\mathrm{d}x\Big)^{1/p}\Big(\int_Q 1^{p'}\,\mathrm{d}x\Big)^{1/p'}\\ &\le m(Q)^{\frac{1}{p'}-1}\Big(\int_Q|f(x)|^p\,\mathrm{d}x\Big)^{1/p}\\ &\le \Big(\fint_Q|f(x)|^p\,\mathrm{d}x\Big)^{1/p}.\end{aligned}$$

Similarly

$$\Big(\fint_Q|f(x)|^p\,\mathrm{d}x\Big)^{1/p}\le C\Big(\fint_Q|f(x)|^q\,\mathrm{d}x\Big)^{1/q},\ q>p.$$

Thus it is natural, to call inequality in which the power on the left hand side is larger the *reverse Hölder inequality.* Reverse Hölder inequalities tell, in general, that a function is more integrable than it first appears. We will need the following deep result of Gehring (1973). We skip the lengthy proof.

Lemma 4.29 (Gehring's lemma)**.** *Suppose that for p, $1 < p < \infty$, there exists $C \geq 1$ such that*

$$\Big(\fint_Q |f(x)|^p \,\mathrm{d}x\Big)^{1/p} \leq C \fint_Q |f(x)| \,\mathrm{d}x$$

for all cubes $Q \subset \mathbf{R}^n$. Then there exists $q > p$ such that

$$\Big(\fint_Q |f(x)|^q \,\mathrm{d}x\Big)^{1/q} \leq C \fint_Q |f(x)| \,\mathrm{d}x$$

for all cubes $Q \subset \mathbf{R}^n$.

Theorem 4.30 (reverse Hölder's inequality)**.** *Suppose that $w \in A_p$, $1 \leq p < \infty$. Then there exists $\delta > 0$ and $C > 0$ s.t.*

$$\Big(\frac{1}{m(Q)} \int_Q w^{1+\delta} \,\mathrm{d}x\Big)^{1/(1+\delta)} \leq \frac{C}{m(Q)} \int_Q w \,\mathrm{d}x$$

for all cubes $Q \subset \mathbf{R}^n$.

Proof. Since $w \in A_p$, we have

$$\frac{1}{m(Q)} \int_Q w \,\mathrm{d}x \Big(\frac{1}{m(Q)} \int_Q w^{1/(1-p)} \,\mathrm{d}x\Big)^{p-1} \leq C.$$

On the other hand Hölder's inequality implies for any measurable $f > 0$ (choose $p = p' = 2$ in (4.14)) that

$$\frac{1}{m(Q)} \int_Q f \,\mathrm{d}x \Big(\frac{1}{m(Q)} \int_Q \frac{1}{f} \,\mathrm{d}x\Big) \geq 1.$$

Then we set $f = w^{1/(p-1)}$ and get

$$1 \leq \frac{1}{m(Q)} \int_Q w^{1/(p-1)} \,\mathrm{d}x \Big(\frac{1}{m(Q)} \int_Q \Big(\frac{1}{w}\Big)^{1/(p-1)} \,\mathrm{d}x\Big).$$

Combining the inequalities for w , we get

$$\begin{aligned}\frac{1}{m(Q)} \int_Q w \,\mathrm{d}x &\Big(\frac{1}{m(Q)} \int_Q w^{1/(1-p)} \,\mathrm{d}x\Big)^{p-1} \\ &\leq \Big(\frac{C}{m(Q)} \int_Q w^{1/(p-1)} \,\mathrm{d}x\Big)^{p-1} \Big(\frac{1}{m(Q)} \int_Q w^{1/(1-p)} \,\mathrm{d}x\Big)^{p-1}.\end{aligned}$$

so that

$$\frac{1}{m(Q)} \int_Q w \,\mathrm{d}x \leq \Big(\frac{C}{m(Q)} \int_Q w^{1/(p-1)} \,\mathrm{d}x\Big)^{p-1}$$

or recalling f

$$\Big(\frac{1}{m(Q)}\int_Q f^{p-1}\,\mathrm{d}x\Big)^{1/(p-1)} \le \frac{C}{m(Q)}\int_Q f\,\mathrm{d}x.$$

Now, we may suppose that $p > 2$ because due to Theorem 4.15, we have $A_p \subset A_q$, $1 \le p < q$, and by this assumption $p-1 > 1$. By Gehring's lemma Lemma 4.29, there exists $q > p-1$ such that

$$\Big(\frac{1}{m(Q)}\int_Q f^{q}\,\mathrm{d}x\Big)^{1/q} \le \frac{C}{m(Q)}\int_Q f\,\mathrm{d}x$$

or again recalling f and taking power $p-1$ on both sides

$$\Big(\frac{1}{m(Q)}\int_Q w^{q/(p-1)}\,\mathrm{d}x\Big)^{(p-1)/q} \le \Big(\frac{C}{m(Q)}\int_Q w^{1/(p-1)}\,\mathrm{d}x\Big)^{p-1}.$$

The right hand side is estimated by using Hölder's inequality as

$$\Big(\frac{1}{m(Q)}\int_Q w^{1/(p-1)}\,\mathrm{d}x\Big)^{p-1} \le \frac{1}{m(Q)}\int_Q w\,\mathrm{d}x$$

and the proof is completed by choosing δ such that $1+\delta = q/(p-1)$. □

Theorem 4.31. *If $w \in A_p$, then $w \in A_{p-\varepsilon}$ for some $\varepsilon > 0$.*

Proof. First we observe that if $w \in A_p$, then (4. Exercise, problem 4)

$$w^{1-p'} \in A_{p'}.$$

Utilizing the previous theorem (Theorem 4.30) for $\left(\frac{1}{w}\right)^{p'-1} = \left(\frac{1}{w}\right)^{1/(p-1)}$, we see that

$$\Big(\frac{1}{m(Q)}\int_Q \Big(\frac{1}{w}\Big)^{(1+\delta)/(p-1)}\,\mathrm{d}x\Big)^{(p-1)/(1+\delta)} \le \Big(\frac{C}{m(Q)}\int_Q \Big(\frac{1}{w}\Big)^{1/(p-1)}\,\mathrm{d}x\Big)^{p-1}.$$

Now we can choose $\varepsilon > 0$ such that

$$\frac{p-1}{1+\delta} = (p-\varepsilon)-1$$

We utilize this and multiply the previous inequality by $\frac{1}{m(Q)}\int_Q w\,\mathrm{d}x$ to have

$$\begin{aligned}\frac{1}{m(Q)}\int_Q w\,\mathrm{d}x&\Big(\frac{1}{m(Q)}\int_Q \Big(\frac{1}{w}\Big)^{1/((p-\varepsilon)-1)}\,\mathrm{d}x\Big)^{(p-\varepsilon)-1}\\ &\le \frac{1}{m(Q)}\int_Q w\,\mathrm{d}x\Big(\frac{C}{m(Q)}\int_Q \Big(\frac{1}{w}\Big)^{1/(p-1)}\,\mathrm{d}x\Big)^{p-1}\\ &\overset{w\in A_p}{\le} C.\end{aligned}$$

Thus $w \in A_{p-\varepsilon}$. □

Next we answer the original question.

Theorem 4.32 (Muckenhoupt)**.** *Let* $1 < p < \infty$*. Then there exists* $C > 0$ *s.t.*

$$\int_{\mathbf{R}^n} \Big(Mf(x)\Big)^p w(x)\,\mathrm{d}x \le C \int_{\mathbf{R}^n} |f(x)|^p\, w(x)\,\mathrm{d}x$$

if and only if $w \in A_p$.

Proof. "$\Rightarrow$" has already been proven.

"$\Leftarrow$" We know that $w > 0$ a.e. so that

$$0 = \mu(E) = \int_E w(x)\,\mathrm{d}x \quad \Leftrightarrow \quad m(E) = 0.$$

and thus

$$\begin{aligned} ||f||_{L^\infty(\mu)} &\overset{\text{def}}{=} \inf\{\lambda \,:\, \mu(\{x \in \mathbf{R}^n \,:\, |f(x)| > \lambda\}) = 0\} \\ &= \inf\{\lambda \,:\, m(\{x \in \mathbf{R}^n \,:\, |f(x)| > \lambda\}) = 0\} \\ &= ||f||_\infty . \end{aligned}$$

Then

$$||Mf||_{L^\infty(\mu)} = ||Mf||_\infty \overset{\text{Lemma 2.8}}{\le} ||f||_\infty = ||f||_{L^\infty(\mu)}$$

so that M is of a weighted strong type (∞, ∞). On the other hand, by Theorem 4.27 implies that M is of weak type (p, p). Moreover, the Marcinkiewicz interpolation theorem Theorem 2.21 holds for all the measures. Thus M is of strong type (q, q) with $q > p$

$$||Mf||_{L^q(\mu)} \le C\, ||f||_{L^q(\mu)} .$$

By the previous theorem $w \in A_p$ implies that $w \in A_{p-\varepsilon}$. Thus we can repeat the above argument starting with $p - \varepsilon$ to see that

$$||Mf||_{L^p(\mu)} \le C\, ||f||_{L^p(\mu)}$$

with the original p. □

7.10.2010

5. Fourier transform

5.1. **On rapidly decreasing functions.** We define a Fourier transform of $f \in L^1(\mathbf{R})$ as

$$F(f) = \hat{f}(\xi) = \int_{\mathbf{R}} f(x) e^{-2\pi i x\xi}\,\mathrm{d}x. \tag{5.1}$$

Remark 5.2. (i) $e^{-2\pi i x\xi} = \cos(2\pi x\xi) - i \sin(2\pi x\xi)$, (even part in real, and odd in imaginary).

(ii) Theory generalizes to $\mathbf{R}^n$ (then $\mathbf{x} \cdot \xi = \sum_{i=1}^n x_i \xi_i$ and $e^{-2\pi i \mathbf{x}\cdot\xi}$).