Harnack's inequality for quasiminimizers with non-standard growth conditions

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Abstract

We prove Harnack inequalities for quasiminimizers of the variable exponent Dirichlet energy integral by employing the De Giorgi method.

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1 Introduction

We study the regularity properties of quasiminimizers of the variable exponent Dirichlet energy integral

$$\int |\nabla u|^{p(x)} \, \mathrm{d}x. \tag{1.1}$$

Here $p(\cdot)$ is a positive, continuous and bounded function which is also bounded away from one. A quasiminimizer minimizes a variational integral up to a multiplicative constant $K \geq 1$, and minimizers are included in the definition as

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the case $K = 1$. Quasiminizers were apparently first studied by Giaquinta – Giusti [13,14]; see also [23,24,27]. Here we adapt the argument of DiBenedetto – Trudinger [6] to the integral (1.1). In the variable exponent setting, the class of quasiminimizers is more flexible since constant multiples of quasiminimizers are still quasiminimizers. As far as we know, this property has not been previously exploited.

The variational integral (1.1) exhibits growth of '$p(x)$-type', which is a particular class of so called nonstandard growth conditions. There is an extensive literature on the calculus of variations and partial differential equations with various nonstandard growth conditions, see for example [1,2,20], and the references in the survey [21]. In particular, quasiminimizers of (1.1) have been studied by Fan – Zhao [8,9,11] and Chiadò Piat – Coscia [5]. However, these authors consider only Hölder continuity, not Harnack's inequality.

Harnack's inequality and other regularity results for (1.1) require additional assumptions on the function $p(\cdot)$; see the counterexamples in [17,26]. The so called logarithmic Hölder continuity condition seems to be the right one for our purposes. This condition was originally introduced by Zhikov [25] in the context of the Lavrentiev phenomenon for minimizers of (1.1), and it has turned out to be a useful tool in regularity and other applications, see, e.g., [2,3,7,8,22].

We start by adapting a standard Caccioppoli type estimate for quasiminimizers of (1.1). Then we use the Caccioppoli estimate to show that for a fixed, non-negative quasiminimizer u, the inequality

$$\sup_{x\in Q_R} u(x) \le C(\inf_{x\in Q_R} u(x) + R) \tag{1.2}$$

holds with a constant depending on L^t norms of u for small t, provided that the cube Q_R is sufficiently small. This estimate is similar to the ones previously obtained for solutions of the Euler-Lagrange equation of (1.1) by Moser's iteration, [3,16]. Note that it is not known whether Moser's iteration works for quasiminimizers, even with a constant exponent, [4]. Hölder continuity is not used in the proof of (1.2). Further, the quasiminimizing property is only used to prove the Caccioppoli estimate. Hence one can say that (1.2) holds for functions belonging to an appropriately defined De Giorgi class.

The features of (1.2) are essentially due to passing between a constant exponent in the standard Sobolev inequality to a variable exponent in the Caccioppoli estimate and back. The first of these steps results in the scale term R on the right hand side of (1.2), and the second in a constant depending on the quasiminimizer u. The necessity of a scale term is not known, but the dependency on u cannot be avoided in the variable exponent case. The latter fact can be shown by considering examples on the real line, [16, Example 3.10].

In the last part, we apply the scaling property of quasiminimizers. The quasiminimizing property can be used to to estimate a quasiminimizer u by estimating u/R^{α}, $\alpha > 0$, in a cube with side length comparable to R. Hence we can refine Harnack's inequality to the form

$$\sup_{x\in Q_R} u(x) \leq C(\inf_{x\in Q_R} u(x) + R^{1+\alpha}) \tag{1.3}$$

for any $\alpha > 0$ with a constant depending on α and the supremum of the original quasiminimizer u. This dependency is due to certain technical adjustments, which we need to handle the scaling.

2 The spaces $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$

Let $p\colon \mathbf{R}^n \to (1,\infty)$ be a bounded measurable function, called variable exponent. Let Ω be an open, bounded subset of $\mathbf{R}^n, n \geq 2$. We denote by $Q(x,r)$ a cube with a center x, side length r and sides parallel to the coordinate axes. Usually we drop the center and write Q_r.

The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ consists of all measurable functions u defined on Ω for which the $p(\cdot)$-modular

$$\varrho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x$$

is finite. The Luxemburg norm on this space is defined as

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\Big\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)}\,\mathrm{d}x \leq 1\Big\}.$$

Equipped with this norm $L^{p(\cdot)}(\Omega)$ is a Banach space. For basic results on variable exponent spaces, we refer to [18].

The modular and the Luxemburg norm are related by the following inequalities, see [10, Theorem 2.1].

$$\begin{aligned}
&\|u\|_{L^{p(\cdot)}(\Omega)} = 1 \text{ if and only if } \varrho_{p(\cdot)}(u) = 1;\\
&\text{if } \|u\|_{L^{p(\cdot)}(\Omega)} > 1, \text{ then } \|u\|^{\inf p}_{L^{p(\cdot)}(\Omega)} \leq \varrho_{p(\cdot)}(u) \leq \|u\|^{\sup p}_{L^{p(\cdot)}(\Omega)};\\
&\text{if } \|u\|_{L^{p(\cdot)}(\Omega)} < 1, \text{ then } \|u\|^{\sup p}_{L^{p(\cdot)}(\Omega)} \leq \varrho_{p(\cdot)}(u) \leq \|u\|^{\inf p}_{L^{p(\cdot)}(\Omega)}.
\end{aligned} \tag{2.1}$$

A version of Hölder's inequality,

$$\int_\Omega fg\,\mathrm{d}x \leq C\|f\|_{L^{p(\cdot)}(\Omega)}\|g\|_{L^{p'(\cdot)}(\Omega)}, \tag{2.2}$$

holds for functions $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$, where the conjugate exponent $p'(\cdot)$ of $p(\cdot)$ is defined pointwise.

The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ consists of functions $u \in L^{p(\cdot)}(\Omega)$ whose distributional gradient ∇u exists almost everywhere and belongs to $L^{p(\cdot)}(\Omega)$. The space $W^{1,p(\cdot)}(\Omega)$ is a Banach space with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

The local Sobolev space $W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega)$ is defined in an analogous way.

An exponent $p(\cdot)$ is said to be log-Hölder continuous if

$$|p(x) - p(y)| \le \frac{C}{-\log(|x-y|)} \tag{2.3}$$

for all $x, y \in \mathbf{R}^n$ such that $|x-y| \le 1/2$. If $1 < \inf p \le \sup p < \infty$ and $p(\cdot)$ is log-Hölder continuous, then the maximal operator is locally bounded and smooth functions are dense in Sobolev space, see [7,22]. Density allows us to pass from smooth test functions to Sobolev test functions by the usual approximation argument. We will use the logarithmic Hölder continuity in the form

$$R^{-(\sup p - \inf p)} \le C, \tag{2.4}$$

where the infimum and the supremum is taken over a cube with side length R. Requiring (2.4) to hold for all cubes is equivalent with condition (2.3), as shown in [7].

From now on, we assume that $1 < \inf p \le \sup p < \infty$ *and* $p(\cdot)$ *is* log-*Hölder continuous.*

3 Quasiminimizers

A function $u \in W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega)$ is called a K-quasiminimizer, *quasiminimizer* for short, if there exists a constant $K \ge 1$ so that for every open set $D \Subset \Omega$ and for every $v \in W^{1,p(\cdot)}(D)$ with compact support in D we have

$$\int_{v \neq 0} |\nabla u|^{p(x)} \,\mathrm{d}x \le K \int_{v \neq 0} |\nabla (u+v)|^{p(x)} \,\mathrm{d}x.$$

If $K = 1$, then u is a minimizer.

We observe that if u is a quasiminimizer with a constant K, then $-u$ is also a quasiminimizer with the same constant, and if $\alpha, \beta \in \mathbf{R}$ then $u + \beta$ and αu are quasiminimizers with constants K and $\max(\alpha^{\sup p - \inf p} K, \alpha^{\inf p - \sup p} K)$, respectively. For log-Hölder continuous exponents, we have the following local version of this property.

Lemma 3.1 *Let $\alpha > 0$. If u is a quasiminimizer with constant K, then u/R^α is quasiminimizer in any cube $Q_{4R} = Q(x_0, 4R)$ with constant $C^{2\alpha}K$, where C depends only on the constant of inequality* (2.4).

Proof. Let x and y be any points in the cube $Q_{4R} = Q(x_0, 4R)$. The inequalities

$$C^{-\alpha} R^{-\alpha p(y)} \leq R^{-\alpha p(x)} \leq C^{\alpha} R^{-\alpha p(y)} \tag{3.2}$$

are elementary consequences of (2.4). Let $v \in W^{1,p(\cdot)}(Q_{4R})$ be compactly supported in Q_{4R}. We use (3.2) and the quasiminimizing property of u, and infer that

$$\begin{aligned} K \int_{v\neq 0} |\nabla\left((u+v)/R^\alpha\right)|^{p(x)}\,\mathrm{d}x &\geq KC^{-\alpha}R^{-\alpha p(y)} \int_{v\neq 0} |\nabla\left(u+v\right)|^{p(x)}\,\mathrm{d}x \\ &\geq C^{-\alpha}R^{-\alpha p(y)} \int_{v\neq 0} |\nabla u|^{p(x)}\,\mathrm{d}x \geq C^{-2\alpha} \int_{v\neq 0} |\nabla(u/R^\alpha)|^{p(x)}\,\mathrm{d}x, \end{aligned}$$

from which the claim follows. $\square$

We will use the following technical lemma.

Lemma 3.3 *Suppose that there is $0 < \delta < 1$, $g \in L^{p(\cdot)}(\Omega)$ and $A > 0$ such that*

$$f(s) \leq \delta f(r) + A \int_\Omega \left(\frac{g(x)}{r-s}\right)^{p(x)} dx$$

for every $\sigma \leq s < r \leq \rho$. Then there exists a constant $C = C(\delta, A, \sup p)$ such that

$$f(\sigma) \leq C \int_\Omega \left(\frac{g(x)}{\rho-\sigma}\right)^{p(x)} dx.$$

We refer to [12, Lemma 3.1, p. 161] for the proof with a constant exponent. The variable exponent case is an easy adaptation of that proof.

We denote the level sets of u by

$$A(k, x_0, r) = A(k, r) = \{x \in Q(x_0, r) : u \geq k\},$$

where $k \in \mathbf{R}$, $x_0 \in \mathbf{R}^n$ and $r > 0$. We also denote $v_+ = \max(v, 0)$.

The following Caccioppoli estimate is well-known, see [8,9,11]. We present the proof here for completeness.

Lemma 3.4 *Let u be a quasiminimizer in Ω. Then there is a constant $C =$*

$C(n, \sup p, K)$ *such that*

$$\int_{A(k,\sigma)} |\nabla u|^{p(x)}\, dx \le C \int_{A(k,\tau)} \left(\frac{(u-k)_+}{\tau - \sigma}\right)^{p(x)} dx$$

for every $0 < \sigma < \tau < R$ *and* $x_0 \in \Omega$ *with* $Q(x_0, R) \Subset \Omega$.

Proof. Let $0 < \sigma \le s < t \le \tau < R$. Let $\eta \in C_0^\infty(Q(x_0, t))$ be a cut-off function such that $0 \le \eta \le 1$, $\eta = 1$ in $Q(x_0, s)$, and $|\nabla \eta| \le \frac{C}{t-s}$.

We choose the test function $v = u - \eta(u-k)_+$. Clearly $v \in W^{1,p(\cdot)}_{\mathrm{loc}}(Q(x_0, R))$ and $A(k, s) \subset \operatorname{spt}(u - v) \subset A(k, t)$, and in $A(k, t)$ we have

$$\nabla v = (1-\eta)\nabla u - (u-k)_+ \nabla \eta.$$

It follows from the quasiminimizing property that

$$\int_{A(k,s)} |\nabla u|^{p(x)}\, \mathrm{d}x \le K \int_{A(k,t)} |\nabla v|^{p(x)}\, \mathrm{d}x.$$

In $A(k, t)$ we obtain the estimate

$$|\nabla v|^{p(x)} \le C\Big((1-\eta)^{p(x)} |\nabla u|^{p(x)} + \Big(\frac{(u-k)_+}{t-s}\Big)^{p(x)}\Big).$$

Since $\eta = 1$ in $A(k, s)$ we conclude

$$\int_{A(k,s)} |\nabla u|^{p(x)}\, \mathrm{d}x \le CK \int_{A(k,t)\setminus A(k,s)} |\nabla u|^{p(x)}\, \mathrm{d}x + CK \int_{A(k,t)} \Big(\frac{(u-k)_+}{t-s}\Big)^{p(x)} \mathrm{d}x.$$

By adding the term $CK \int_{A(k,s)} |\nabla u|^{p(x)}\, \mathrm{d}x$ to both sides, it follows that

$$\begin{aligned}\int_{A(k,s)} |\nabla u|^{p(x)}\, \mathrm{d}x \le& \frac{CK}{CK+1} \int_{A(k,t)} |\nabla u|^{p(x)}\, \mathrm{d}x \\ &+ \frac{CK}{CK+1} \int_{A(k,t)} \Big(\frac{(u-k)_+}{t-s}\Big)^{p(x)} \mathrm{d}x.\end{aligned}$$

Now an application of Lemma 3.3 concludes the proof. □

The quasiminimizing property is not used in the next two sections. Hence one could define the De Giorgi class $\mathrm{DG}_{p(\cdot)}(\Omega)$ to consist of functions $u \in W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega)$ such that u and $-u$ satisfy the Caccioppoli estimate, and then say that the results in Sections 4 and 5 hold true for functions $u \in \mathrm{DG}_{p(\cdot)}(\Omega)$.

4 Local boundedness

This section concentrates on the local boundedness of quasiminimizers. First, we show that it is possible to obtain an estimate for the essential supremum without the additional R on the right hand side (cf. [3,16]). In particular, this version yields the classical estimate for a constant $p(\cdot)$. At the end of the section, we derive a version with the additional R. This version appears to be more useful when proving the weak Harnack inequality.

We work in a cube $Q = Q_{2R} \Subset \Omega$ and denote

$$p^+ = \sup_{x \in Q} p(x), \qquad p^- = \inf_{x \in Q} p(x).$$

Our key tool is the next lemma from [16, Lemma 3.4]. We apply it to match the different exponents emerging from Caccioppoli and Sobolev type inequalities.

Lemma 4.1 *Let f be a positive measurable function and assume that the exponent $p(\cdot)$ is* log-*Hölder continuous. Then*

$$\fint_Q f^{p^+ - p^-}\, dx \le C \|f\|_{L^s(Q)}^{p^+ - p^-}$$

for any $s > p^+ - p^-$, where the constant depends on n, $p(\cdot)$ and s.

Before proving the first De Giorgi estimate, we define some quantities used throughout the paper. We pick an exponent $q > 1$ to be used in connection with Lemma 4.1. To achieve the necessary disparity of level sets in the next lemma, we would like the quantity $1 - qp^-/(p^-)^*$ to be positive. Here $(p^-)^*$ is the Sobolev conjugate of p^-. To ensure the positivity, we require that

$$1 < q < \frac{n}{n-1}.$$

We will take $f = u^{q'}$ in Lemma 4.1. In this case, the upper bound in terms of u is

$$\|u\|_{L^{q's}(Q)}^{q'(p^+ - p^-)}.$$

By choosing s such that $q's = p^-$, we see that all constants in the estimates are finite. Due to the continuity of $p(\cdot)$, such a choice of s is possible in spite of the requirement $s > p^+ - p^-$ if we choose $R \le 1$ small enough. Further, we choose the cube Q to be small enough so that

$$\int_Q |u|^{p(x)}\, \mathrm{d}x \le 1 \text{ and } \int_Q |\nabla u|^{p(x)}\, \mathrm{d}x \le 1. \tag{4.2}$$

This assumption allows us to employ the modular inequality (2.1) together with Hölder's inequality (2.2), and avoid the additional scale term R in the estimate.

Lemma 4.3 *Let u be a quasiminimizer and let q, s and Q be as above. Then for $\varepsilon = 1 - qp^-/(p^-)^*$, all $0 \le h < k$ and $R/2 \le \sigma < \tau \le R \le 1$, we have the estimate*

$$\fint_{Q_\sigma} (u-k)_+^{qp^-}\, dx \le C\Big(\fint_{Q_\tau} (u-h)_+^{qp^-}\, dx\Big)^{p^-/p^+ + \varepsilon} \frac{1}{(k-h)^{qp^- \varepsilon}} \frac{\tau^{qp^-}}{(\tau-\sigma)^{qp^-}}.$$

The constant C depends on n, $p(\cdot)$, q, K, and the $L^{q's}(Q)$-norm of u.

Proof. Let $\eta \in C_0^\infty(Q_\tau)$ be a cut-off function such that $0 \le \eta \le 1$, $\eta = 1$ in $Q_{(\sigma+\tau)/2}$ and $|\nabla \eta| \le \frac{C}{\tau-\sigma}$. First, Hölder's inequality implies

$$\begin{aligned}\fint_{Q_\sigma} (u-k)_+^{qp^-}\, \mathrm{d}x \le& C\fint_{Q_{(\sigma+\tau)/2}} \eta^{qp^-} (u-k)_+^{qp^-}\, \mathrm{d}x \\ \le& C\Big(\frac{\big|A(k,\tau)\big|}{\big|Q_{(\sigma+\tau)/2}\big|}\Big)^{1-qp^-/(p^-)^*} \Big(\fint_{Q_{(\sigma+\tau)/2}} \eta^{(p^-)^*}(u-k)_+^{(p^-)^*}\, \mathrm{d}x\Big)^{qp^-/(p^-)^*}.\end{aligned}$$

Then we apply Sobolev's inequality and obtain

$$\fint_{Q_\sigma} (u-k)_+^{qp^-}\, \mathrm{d}x \le C\Big(\frac{\big|A(k,\tau)\big|}{\big|Q_{(\sigma+\tau)/2}\big|}\Big)^{1-qp^-/(p^-)^*} \tau^{qp^-} \Big(\fint_{Q_{(\sigma+\tau)/2}} |\nabla(\eta(u-k)_+)|^{p^-}\, \mathrm{d}x\Big)^q.$$

Furthermore, we have

$$\begin{aligned}&\fint_{Q_{(\sigma+\tau)/2}} |\nabla(\eta(u-k)_+)|^{p^-}\, \mathrm{d}x \\ &\qquad\le C\fint_{Q_\tau} |\nabla \eta|^{p^-} (u-k)_+^{p^-}\, \mathrm{d}x + \fint_{Q_{(\sigma+\tau)/2}} |\nabla (u-k)_+|^{p^-}\, \mathrm{d}x.\end{aligned} \tag{4.4}$$

Below, we want to estimate $\fint_{Q_\tau}(u-k)_+^{p(x)}\, \mathrm{d}x$ by $\fint_{Q_\tau}(u-k)_+^{qp^-}\, \mathrm{d}x$. We accomplish this by first using Hölder's inequality. This leads to

$$\fint_{Q_\tau} (u-k)_+^{p(x)}\, \mathrm{d}x \le \Big(\fint_{Q_\tau} (u-k)_+^{q'(p(x)-p^-)}\, \mathrm{d}x\Big)^{1/q'} \Big(\fint_{Q_\tau} (u-k)_+^{qp^-}\, \mathrm{d}x\Big)^{1/q}.$$

Then we estimate the first integral by using Lemma 4.1, and obtain

$$\fint_{Q_\tau} (u-k)_+^{q'(p(x)-p^-)}\, \mathrm{d}x \le C\fint_Q (1+|u|^{q'(p^+-p^-)})\, \mathrm{d}x \le C(1+\|u\|_{L^{q's}(Q)}^{q'(p^+-p^-)})^{q'}.$$

Thus, we have

$$\fint_{Q_\tau} (u-k)_+^{p(x)}\, \mathrm{d}x \le C\Big(\fint_{Q_\tau} (u-k)_+^{qp^-}\, \mathrm{d}x\Big)^{1/q}, \tag{4.5}$$

where $C = \widetilde{C}(1+\|u\|_{L^{q's}(Q)}^{(p^+-p^-)})$, and $\widetilde{C}$ is a constant independent of u.

The next step is to pass from p^- to $p(x)$ in order to use the Caccioppoli estimate, Lemma 3.4. To this end, we use Hölder's inequality (2.2) and the modular inequality (2.1). We obtain

$$
\begin{aligned}
\fint_{Q_{(\sigma+\tau)/2}} |\nabla(u-k)_+|^{p^-}\,\mathrm{d}x \le{}& \frac{C}{|Q_{(\sigma+\tau)/2}|}\|1\|_{(p(\cdot)/p^-)'}\||\nabla(u-k)_+|^{p^-}\|_{p(\cdot)/p^-} \\
\le{}& C|Q_{(\sigma+\tau)/2}|^{-1+p^-/p^+}\left(\fint_{Q_{(\sigma+\tau)/2}} |\nabla(u-k)_+|^{p(x)}\,\mathrm{d}x\right)^{p^-/p^+} \\
\le{}& C\left(\fint_{Q_{(\sigma+\tau)/2}} |\nabla(u-k)_+|^{p(x)}\,\mathrm{d}x\right)^{p^-/p^+} \\
\le{}& C\left(\fint_{Q_\tau}\left(\frac{(u-k)_+}{\tau-\sigma}\right)^{p(x)}\mathrm{d}x\right)^{p^-/p^+} \\
\le{}& \frac{C}{(\tau-\sigma)^{p^-}}\left(\fint_{Q_\tau}((u-k)_+)^{p(x)}\,\mathrm{d}x\right)^{p^-/p^+} \\
\le{}& \frac{C}{(\tau-\sigma)^{p^-}}\left(\fint_{Q_\tau}((u-k)_+)^{qp^-}\,\mathrm{d}x\right)^{p^-/qp^+}.
\end{aligned}
$$

We also used logarithmic Hölder continuity (2.4) in the third inequality, and (4.5) in the last inequality.

Similarly, we have

$$
\begin{aligned}
\fint_{Q_\tau} |\nabla\eta|^{p^-}(u-k)_+^{p^-}\,\mathrm{d}x \le{}& \frac{C}{(\tau-\sigma)^{p^-}}\fint_{Q_\tau}(u-k)_+^{p^-}\,\mathrm{d}x \\
\le{}& \frac{C}{(\tau-\sigma)^{p^-}}|Q_\tau|^{-1}\|1\|_{(p(\cdot)/p^-)'}\|(u-k)_+^{p^-}\|_{p(\cdot)/p^-} \\
\le{}& \frac{C}{(\tau-\sigma)^{p^-}}\left(\fint_{Q_\tau}(u-k)_+^{p(x)}\,\mathrm{d}x\right)^{p^-/p^+} \\
\le{}& \frac{C}{(\tau-\sigma)^{p^-}}\left(\fint_{Q_\tau}(u-k)_+^{qp^-}\,\mathrm{d}x\right)^{p^-/qp^+}.
\end{aligned}
$$

Collecting the estimates, we obtain

$$
\fint_{Q_\sigma}(u-k)_+^{qp^-}\,\mathrm{d}x \le C\left(\frac{|A(k,\tau)|}{|Q_{(\sigma+\tau)/2}|}\right)^{1-qp^-/(p^-)^*}\frac{\tau^{qp^-}}{(\tau-\sigma)^{qp^-}}\left(\fint_{Q_\tau}(u-k)_+^{qp^-}\,\mathrm{d}x\right)^{p^-/p^+}.
$$

Finally, we observe that

$$
\frac{|A(k,\tau)|}{|Q_{(\sigma+\tau)/2}|} \le C\frac{|A(k,\tau)|}{|Q_\tau|} \le \frac{C}{(k-h)^{qp^-}}\fint_{Q_\tau}(u-h)_+^{qp^-}\,\mathrm{d}x
$$

and

$$\fint_{Q_\tau} (u-k)_+^{qp^-}\,\mathrm{d}x \le \fint_{Q_\tau} (u-h)_+^{qp^-}\,\mathrm{d}x$$

for every $h < k$. We recall that $\varepsilon = 1 - qp^-/(p^-)^* > 0$, and combine the last two estimates with (4). It follows that

$$\fint_{Q_\sigma} (u-k)_+^{qp^-}\,\mathrm{d}x \le C\Big(\fint_{Q_\tau} (u-h)_+^{qp^-}\,\mathrm{d}x\Big)^{p^-/p^+ + \varepsilon} \frac{1}{(k-h)^{qp^-\varepsilon}} \frac{\tau^{qp^-}}{(\tau-\sigma)^{qp^-}},$$

and the proof is complete. $\square$

The following iteration lemma turns out to be useful in proving the local boundedness for a quasiminimizer. See, e.g., [15, Lemma 7.1] for the proof.

Lemma 4.6 *Let $\varepsilon > 0$ and suppose that (ω_i) is a sequence of real numbers such that*

$$\omega_{i+1} \le CL^i \omega_i^{1+\varepsilon}$$

with $C > 0$ and $L > 1$. If $\omega_0 \le C^{-1/\varepsilon} L^{-1/\varepsilon^2}$, then we have

$$\omega_i \le L^{-i/\varepsilon}\omega_0$$

and in particular, $\omega_i \to 0$ $i \to \infty$.

Theorem 4.7 *Let u be a quasiminimizer, $k_0 \in \mathbf{R}$ and let Q, R, q and s be as defined in the beginning of the section. Then we have the estimate*

$$\operatorname*{ess\,sup}_{Q_{R/2}} u \le k_0 + C\left(\fint_{Q_R} (u-k_0)_+^{qp^-}\,dx\right)^{\delta/(\varepsilon q p^-)}, \tag{4.8}$$

where $\varepsilon = 1 - qp^-/(p^-)^*$ and $\delta = p^-/p^+ + \varepsilon - 1$. The constant C depends on n, $p(\cdot)$, q, K and the $L^{q's}(Q)$-norm of u.

Proof. Let $d \ge 0$ be a number to be fixed later. Define for $i = 1, 2, \dots$ that

$$k_i = k_0 + d(1 - 2^{-i})$$

and

$$\sigma_i = \frac{R}{2}(1 + 2^{-i}).$$

It follows that $\lim_{i\to\infty} k_i = k_0 + d$, $\sigma_0 = R$ and $\lim_{i\to\infty}\sigma_i = R/2$. We set $\sigma = \sigma_{i+1}$, $\tau = \sigma_i$, $k = k_{i+1}$ and $h = k_i$ in Lemma 4.3. This provides the estimate

$$\fint_{Q_{\sigma_{i+1}}} (u-k_{i+1})_+^{qp^-}\,\mathrm{d}x \le \frac{C}{d^{qp^-\varepsilon}2^{-i\varepsilon qp^-}} \frac{1}{2^{-iqp^-}} \Big(\fint_{Q_{\sigma_i}} (u-k_i)_+^{qp^-}\,\mathrm{d}x\Big)^{1+\delta}. \tag{4.9}$$

Furthermore, if we define

$$\Phi_i = d^{-qp^-} \fint_{Q_{\sigma_i}} (u-k_i)_+^{qp^-}\,\mathrm{d}x,$$

then (4.9) becomes

$$\Phi_{i+1} \leq C 2^{iqp^-(1+\varepsilon)} d^{-qp^-(\varepsilon-\delta)} \Phi_i^{1+\delta}.$$

Next we utilize Lemma 4.6. Here $L = 2^{qp^-(1+\varepsilon)}$, and the condition in Lemma 4.6 becomes

$$\Phi_0 = d^{-qp^-} \fint_{Q_R} (u - k_0)_+^{qp^-} \, \mathrm{d}x \leq (C d^{-qp^-(\varepsilon-\delta)})^{-1/\delta} 2^{-qp^-(1+\varepsilon)/\delta^2}. \tag{4.10}$$

The condition (4.10) is satisfied if we choose

$$d = C \left(\fint_{Q_R} (u - k_0)_+^{qp^-} \, \mathrm{d}x \right)^{\delta/(\varepsilon q p^-)},$$

and it follows from Lemma 4.6 that

$$\lim_{i \to \infty} \Phi_i = 0.$$

Consequently,

$$\operatorname*{ess\,sup}_{Q_{R/2}} u \leq k_0 + d \leq k_0 + C \left(\fint_{Q_R} (u - k_0)_+^{qp^-} \, \mathrm{d}x \right)^{\delta/(\varepsilon q p^-)},$$

which is the desired assertion. □

Instead of the cube $Q_{R/2}$, it is possible to have any smaller cube Q_{tR}, $t < 1$, on the left hand side in the previous estimate. Further, we can have any small positive power on the right hand side. The arguments used to establish these facts are standard; see, e.g., [15, Corollary 7.1 and Theorem 7.3].

Corollary 4.11 *With the assumptions of the previous theorem, we have for any $t < 1$ the estimate*

$$\operatorname*{ess\,sup}_{Q_{tR}} u \leq k_0 + C \left(\frac{1}{(1-t)^n} \fint_{Q_R} (u - k_0)_+^{qp^-} \, dx \right)^{\delta/(\varepsilon q p^-)}, \tag{4.12}$$

where δ and ε are as in Theorem 4.7. The constant C depends on n, $p(\cdot)$, q, K and the $L^{q's}(Q)$-norm of u.

Theorem 4.13 *Let u be a quasiminimizer and let Q, R, q, s, ε and δ be as in Theorem 4.7. Then for every $l \in (0, qp^-)$ and $\rho < R$, we have the estimate*

$$\operatorname*{ess\,sup}_{Q_\rho} u \leq k_0 + C \left(\frac{1}{(R-\rho)^n} \int_{Q_R} (u - k_0)_+^{l} \, dx \right)^{\delta/((\varepsilon-\delta)qp^- + l\delta)}.$$

The constant C depends on l, n, $p(\cdot)$, q, K and the $L^{q's}(Q)$-norm of u.

Even though Theorem 4.7 is natural in the sense that it gives the right estimate for a constant $p(\cdot)$, it is hard to utilize it in proving Harnack's inequality. This is due to the inhomogeneity in the exponents. Hence we modify the above estimate slightly.

Corollary 4.14 *Let u be a quasiminimizer and let Q, R, q and s be as defined at the beginning of the section. Then for every $l \in (0, qp^-)$ and $\rho < R$, we have the estimate*

$$\operatorname*{ess\,sup}_{Q_\rho} u \leq k_0 + R + \left(\frac{C}{(R-\rho)^n} \int_{Q_R} (u-k_0)_+^l \, dx \right)^{1/l}. \tag{4.15}$$

The constant C depends on l, n, $p(\cdot)$, q, K and the $L^{q's}(Q)$-norm of u.

Proof. We shall use Theorem 4.13, and denote

$$\alpha = \frac{\delta}{(\varepsilon - \delta) qp^- + l\delta},$$

as well as

$$\beta = \frac{1 - \alpha l}{p^+ - p^-} = \frac{qp^-}{(p^+ - p^-)qp^- + p^+ l \delta}.$$

Furthermore, write

$$1 = R^{\left(-(p^+-p^-)+(p^+-p^-)\right)\beta}.$$

By Theorem 4.13, we obtain

$$\operatorname*{ess\,sup}_{Q_\rho} u \leq k_0 + C \left(\frac{R^{\left(-(p^+-p^-)+(p^+-p^-)\right)\beta}}{(R-\rho)^n} \int_{Q_R} (u-k_0)_+^l \, \mathrm{d}x \right)^{\alpha}.$$

We observe that

$$\frac{1}{\alpha l} = \frac{(\varepsilon - \delta)qp^- + l\delta}{\delta l} > 1,$$

and by Young's inequality

$$\operatorname*{ess\,sup}_{Q_\rho} u \leq k_0 + C \left(\frac{R^{-(p^+-p^-)\beta}}{(R-\rho)^n} \int_{Q_R} (u-k_0)_+^l \, \mathrm{d}x \right)^{1/l} + R.$$

Since $R^{-(p^+-p^-)\beta} \leq C$ by log-Hölder continuity, the proof is complete. □

5 Harnack's inequality

In this section we prove the weak Harnack inequality. We proceed as in DiBenedetto – Trudinger [6]. This together with Corollary 4.14 implies Harnack's inequality for nonnegative quasiminimizers, see Corollary 5.13. We denote

$$D(k, x_0, r) = D(k, r) = \{x \in Q(x_0, r) \, : \, u(x) < k\},$$

and start with some auxiliary estimates.

Lemma 5.1 *Let u be a nonnegative quasiminimizer in Ω. Then there exists a constant $\gamma_0 \in (0,1)$, depending on n, $p(\cdot)$, q, K, and the $L^{q's}(Q)$-norm of u, such that if*

$$|D(\vartheta, R)| \leq \gamma_0 |Q_R|$$

for some $\vartheta > 0$, then

$$\operatorname*{ess\,inf}_{Q_{R/2}} u + R \geq \frac{\vartheta}{2}.$$

Proof. Corollary 4.14 applied to $-u$ with $k_0 = -\vartheta$ and $l = 1$ in (4.15) implies

$$\begin{aligned} \operatorname*{ess\,inf}_{Q_{R/2}} u + R &\geq \vartheta - \frac{C}{R^n} \int_{D(\vartheta,R)} (\vartheta - u)_+ \, \mathrm{d}x \\ &\geq \vartheta - C\vartheta \frac{D(\vartheta, R)}{|Q_R|} \geq \vartheta - C\vartheta\gamma_0. \end{aligned}$$

To finish the proof, we choose $\gamma_0 = (2C)^{-1}$, where the constant C is as in Corollary 4.14. □

The following lemma is an improvement of the preceding one. Observe that the result is nontrivial for a large enough level set ϑ, since μ below does not depend on the level set.

Lemma 5.2 *Suppose that the hypothesis of Lemma 5.1 holds. For every $\gamma \in (0,1)$ there exists a constant $\mu > 0$, depending on γ, n, $p(\cdot)$, q, K, and the $L^{q's}(Q)$-norm of u, such that if*

$$|D(\vartheta, R)| \leq \gamma |Q_R|$$

for some $\vartheta > 0$, then

$$\operatorname*{ess\,inf}_{Q_{R/2}} u + R \geq \mu\vartheta.$$

Proof. Let i_0 be a positive integer to be fixed later. Let us first assume that $\vartheta > 2^{i_0} R$. For $R < h < k < \vartheta$ we set

$$v = \begin{cases} 0, & \text{if} \quad u \geq k, \\ k - u, & \text{if} \quad h < u < k, \\ k - h, & \text{if} \quad u \leq h. \end{cases}$$

Then $v \in W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega)$ and $\nabla v = -\nabla u \chi_{\{h<u<k\}}$ a.e. in Ω. Clearly, $v = 0$ in $Q_R \setminus D(k, R)$, and since $|D(k, R)| \leq \gamma |Q_R|$ we obtain $|Q_R \setminus D(k, R)| \geq (1-\gamma)|Q_R|$.

Hence we may apply Sobolev's inequality

$$\left(\int_{Q_R} v^{n/(n-1)}\,\mathrm{d}x\right)^{(n-1)/n} \leq C\int_{\Delta} |\nabla v|\,\mathrm{d}x,$$

where $\Delta := D(k,R)\setminus D(h,R)$ and C depends on γ and n. We have

$$(k-h)|D(h,R)| = \int_{D(h,R)} v\,\mathrm{d}x \leq |D(h,R)|^{1/n}\left(\int_{Q_R} v^{n/n-1}\,\mathrm{d}x\right)^{(n-1)/n},$$

from which it follows that

$$\begin{aligned}(k-h)|D(h,R)|^{(n-1)/n} &\leq C\int_{\Delta}|\nabla v|\,\mathrm{d}x \\ &\leq C|\Delta|^{1-1/p^-}\left(\int_{D(k,R)} |\nabla v|^{p^-}\,\mathrm{d}x\right)^{1/p^-}. \qquad (5.3)\end{aligned}$$

On the other hand, Caccioppoli estimate gives

$$\begin{aligned}\int_{D(k,R)} |\nabla v|^{p(x)}\,\mathrm{d}x &\leq C\int_{D(k,2R)}\left(\frac{(k-u)_+}{R}\right)^{p(x)}\mathrm{d}x \\ &\leq C(k/R)^{p^+}|D(k,2R)| \\ &\leq Ck^{p^+}R^{n-p^+}. \qquad (5.4)\end{aligned}$$

Here we used the fact that $k > R$. To pass from p^- to $p(x)$ in (5.3), we use Hölder's inequality and (2.1). Remember that R is chosen to be so small that

$$\int_{Q_R} |\nabla v|^{p(x)}\,\mathrm{d}x \leq \int_{Q_R} |\nabla u|^{p(x)}\,\mathrm{d}x \leq 1.$$

This gives us

$$\begin{aligned}\int_{D(k,R)} |\nabla v|^{p^-}\,\mathrm{d}x &\leq C\|1\|_{(p(\cdot)/p^-)'}\||\nabla v|^{p^-}\|_{p(\cdot)/p^-} \\ &\leq C\left(\int_{D(k,R)} |\nabla v|^{p(x)}\,\mathrm{d}x\right)^{p^-/p^+} \\ &\leq Ck^{p^-}R^{np^-/p^+-p^-} \\ &\leq Ck^{p^-}R^{n-p^-}, \qquad (5.5)\end{aligned}$$

where the last inequality follows from the logarithmic Hölder continuity of the exponent. From (5.3) and (5.5), we deduce that

$$\left(\frac{k-h}{k}\right)^{p^-/(p^--1)} |D(h,R)|^{p^-(n-1)/n(p^--1)} \leq CR^{(n-p^-)/(p^--1)}|D(k,R)\setminus D(h,R)|.$$

We then consider the sequence of levels $k_i = \vartheta 2^{-i}$ with the nonnegative natural numbers $i \leq i_0$. We set in the previous estimate $k = k_i$ and $h = k_{i+1}$ and denote $d_i = |D(k_i, R)|$. Since $d_i \geq d_{i_0} = |D(\vartheta 2^{i_0}, R)|$ for all $0 \leq i \leq i_0$, we obtain

$$|D(\vartheta 2^{-i_0}, R)|^{p^-(n-1)/n(p^--1)} \leq CR^{(n-p^-)/(p^--1)}(d_i - d_{i+1}).$$

Therefore, summing over i between zero and $i_0 - 1$, we conclude

$$\begin{aligned} i_0|D(\vartheta 2^{-i_0}, R)|^{p^-(n-1)/n(p^--1)} &\leq CR^{(n-p^-)/(p^--1)}(d_0 - d_{i_0}) \\ &\leq CR^{n+(n-p^-)/(p^--1)}, \end{aligned}$$

or, equivalently,

$$|D(\vartheta 2^{-i_0}, R)| \leq \left(\frac{C}{i_0}\right)^{n(p^--1)/p^-(n-1)} |Q_R|.$$

Finally, we can choose i_0, depending only on γ, n, $p(\cdot)$, q, K, and the $L^{q's}(Q)$-norm of u, such that

$$|D(\vartheta 2^{-i_0}, R)| \leq \gamma_0 |Q_R|,$$

where γ_0 is as in Lemma 5.1. Thus, by Lemma 5.1, we have

$$\operatorname*{ess\,inf}_{Q_{R/2}} u + R \geq \vartheta 2^{-i_0-1}, \qquad \vartheta > 2^{i_0}R.$$

Now consider the two cases $\vartheta > 2^{i_0}R$ and $\vartheta \leq 2^{i_0}R$. In the first case, the claim follows with the constant $\mu = 2^{-i_0-1}$ by the above argument. In the second case, $\vartheta \leq 2^{i_0}R$, the claim readily follows with $\mu = 2^{-i_0}$. This completes the proof, since the constant $\mu = 2^{-i_0-1}$ is admissible in both cases. $\square$

The following covering theorem is due to Krylov and Safonov, see [19]. For the proof, see, e.g., the monograph by Giusti [15].

Lemma 5.6 *Let $E \subset Q_R \subset \mathbf{R}^n$ be a measurable set, and let $0 < \delta < 1$. Moreover, let*

$$E_\delta = \bigcup_{x \in Q_R,\, 0<\rho<R} \{Q(x, 3\rho) \cap Q_R \ :\ |Q(x, 3\rho) \cap E| \geq \delta |Q_\rho|\}.$$

Then either $|E| \geq \delta |Q_R|$, in which case $E_\delta = Q_R$, or

$$|E_\delta| \geq \frac{1}{\delta}|E|.$$

We are ready to prove the weak Harnack inequality for quasiminimizers. We closely follow the argumentation in [15], pp. 239–240.

Theorem 5.7 *Let u be a nonnegative quasiminimizer in Ω. Then there exists an exponent $h > 0$ and a constant C, both depending on n, $p(\cdot)$, q, K, and the $L^{q's}(Q)$-norm of u, such that*

$$\left(\fint_{Q(x_0,R)} u^h\, dx\right)^{1/h} \leq C(\operatorname*{ess\,inf}_{Q(x_0,R/2)} u + R)$$

for every cube $Q(x_0, R)$ for which $Q(x_0, 10R) \subset \Omega$.

Proof. We denote $Q_R = Q(x_0, R)$. By Cavalieri's principle, we have

$$\int_{Q_R} (u+R)^h \,\mathrm{d}x = h\int_0^\infty t^{h-1}|A_t^0|\,\mathrm{d}t \tag{5.8}$$

where $A_t^0 = \{x \in Q_R : u(x) + R > t\}$, $t > 0$. In order to estimate the measure of A_t^0, we fix $0 < \delta < 1$ and $\gamma = 1 - \delta/6^n$ and define sets A_t^i by

$$A_t^i = A(t\mu^i, R) = \{x \in Q_R : u(x) + R > t\mu^i\} \subset Q_R,$$

where μ is the constant in Lemma 5.2 corresponding the constant γ above. Suppose that for some $\rho < R$ and $z \in Q_R$, we have

$$Q(z, 3\rho) \cap Q_R \subset (A_t^i)_\delta, \tag{5.9}$$

where $(\cdot)_\delta$ is defined in Lemma 5.6. By the definition of $(A_t^i)_\delta$ and (5.9) it follows that

$$\begin{aligned}\frac{\delta}{6^n}|Q_{6\rho}| = \delta|Q_\rho| &\leq |Q(z,3\rho) \cap A(t\mu^i, x_0, R)| \\ &= |Q_R \cap A(t\mu^i, z, 3\rho)| \leq |A(t\mu^i, z, 6\rho)|.\end{aligned}$$

This estimate implies that

$$|D(t\mu^i, z, 6\rho)| \leq (1 - \frac{\delta}{6^n})|Q_{6\rho}| = \gamma|Q_{6\rho}|. \tag{5.10}$$

The assumption $Q(x_0, 10R) \subset \Omega$ guarantees that also $Q(z, 6\rho) \subset \Omega$. Hence, Lemma 5.2 gives us

$$\operatorname*{ess\,inf}_{Q(z,3\rho)} u + R \geq t\mu^{i+1}. \tag{5.11}$$

Since assumption (5.9) leads to (5.11), it follows that

$$(A_t^i)_\delta \subset A_t^{i+1}.$$

This together with Lemma 5.6 imply that either $A_t^{i+1} = Q_R$ or $|A_t^{i+1}| \geq |(A_t^i)_\delta| \geq \delta^{-1}|A_t^i|$. In any case, if for some positive integer j we have

$$\delta^j|Q_R| \leq |A_t^0| \leq \delta^{j-1}|Q_R|. \tag{5.12}$$

It follows that

$$|A_t^j| \geq \delta^{-1}|A_t^{j-1}| \geq \delta^{-2}|A_t^{j-2}| \geq \ldots \geq \delta^{1-j}|A_t^0| \geq \delta|Q_R|.$$

Therefore $A_t^{j+1} = Q_R$, and, consequently,

$$\operatorname*{ess\,inf}_{Q_{R/2}} u + R \geq t\mu^{j+1}.$$

We choose j so that (5.12) is satisfied. Let, for instance, j be the smallest integer satisfying

$$j \geq \frac{1}{\log \delta} \log \frac{|A_t^0|}{|Q_R|}.$$

With this choice of j, we obtain

$$\operatorname*{ess\,inf}_{Q_{R/2}} u + R \geq t\mu^{j+1} = Ct \left(\frac{|A_t^0|}{|Q_R|} \right)^{\log\mu/\log\delta}$$

or, equivalently, by setting $\xi = \operatorname{ess\,inf}_{Q_{R/2}} u + R$ and $a = \frac{\log\delta}{\log\mu} > 0$, we get

$$|A_t^0| \leq C|Q_R|\xi^a t^{-a}.$$

We choose $0 < h < a$ and obtain by (5.8) that

$$\int_{Q_R} (u+R)^h \,\mathrm{d}x \leq C|Q_R| \left(\xi^h + \xi^a \int_\xi^\infty t^{h-a-1} \,\mathrm{d}t \right) = C|Q_R|\xi^h,$$

which is

$$\left(\frac{1}{|Q_R|} \int_{Q_R} (u+R)^h \,\mathrm{d}x \right)^{1/h} \leq C \operatorname*{ess\,inf}_{Q_{R/2}} u + R.$$

A trivial estimate now completes the proof. □

Corollary 4.14 and Theorem 5.7 imply the following Harnack inequality.

Corollary 5.13 *Let u be a nonnegative quasiminimizer in Ω. Then there exists a constant C such that*

$$\operatorname*{ess\,sup}_{Q(x_0,R)} u \leq C(\operatorname*{ess\,inf}_{Q(x_0,R)} u + R)$$

for every cube $Q(x_0,R)$ for which $Q(x_0,10R) \subset \Omega$. The constant C depends on n, $p(\cdot)$, q, K and the $L^{q's}(Q)$-norm of u.

6 Variant estimates

In this section, we use the scaling property of quasiminimizers to prove a different form of Harnack's inequality. More precisely, we use estimates for u/R^α

and Lemma 3.1 to get estimates for u. This way, one can replace the scale term R by $R^{1+\alpha}$ for any $\alpha > 0$. Some modifications to the estimates of the preceding two sections are required. For instance, the assumptions $\fint_Q |v|^{p(x)}\,\mathrm{d}x \le 1$ and $\fint_Q |\nabla v|^{p(x)}\,\mathrm{d}x \le 1$ do not hold for small values of R when $v = u/R^\alpha$.

We start by observing that the Caccioppoli estimate of Lemma 3.4 holds for u/R^α with a constant independent of R by Lemma 3.1. In order to obtain a suitable version of Lemma 4.3, we use a harsh estimate.

Lemma 6.1 *Let u be a quasiminimizer, choose Q so small that we can take $q's = p^-$ with $s > p^+ - p^-$, and denote $\sigma_i = \frac{R}{2}(1+2^{-i})$. Then*

$$\begin{aligned}\fint_{Q_{\sigma_{i+1}}} (u-k)_+^{qp^-}\, dx \le & C 2^{iqp^+} \left(\fint_{Q_{\sigma_i}} (u-h)_+^{qp^-}\, dx\right)^{1+\varepsilon} \\ & \times \frac{1}{(k-h)^{qp^- \varepsilon}} \left(\frac{R^{qp^-}}{(k-h)^{qp^-}} + 1\right),\end{aligned}$$

where the constant C depends on n, $p(\cdot)$, q, K, and the $L^{q's}(Q)$-norm of u.

Proof. We argue as in Lemma 4.3 up to the inequality (4.4). After this, we apply the harsh estimates

$$\begin{aligned}|\nabla(u-k)_+|^{p^-} \le & (|\nabla(u-k)_+|+1)^{p^-} \le C|\nabla(u-k)_+|^{p(x)} + C \text{ and} \\ |\nabla\eta(u-k)_+|^{p^-} \le & (|\nabla\eta(u-k)_+|+1)^{p^-} \le C|\nabla\eta(u-k)_+|^{p(x)} + C\end{aligned}$$

to the right hand side of (4.4). This, together with the Caccioppoli estimate, leads to the inequality

$$\fint_{Q_{(\sigma+\tau)/2}} |\nabla(\eta(u-k)_+)|^{p^-}\,\mathrm{d}x \le C\left(\fint_{Q_\tau}\left(\frac{(u-k)_+}{\sigma-\tau}\right)^{p(x)}\mathrm{d}x + \frac{|A(k,\tau)|}{|Q_\tau|}\right).$$

We write $\sigma_i = \frac{R}{2}(1+2^{-i})$ and take $\sigma = \sigma_{i+1}$ and $\tau = \sigma_i$. Now we can use log-Hölder continuity to estimate

$$(\sigma_i - \sigma_{i+1})^{-p(x)} \le 2^{ip^+} R^{-p(x)} \le C 2^{ip^+} R^{-p^-}.$$

An application of (4.5) gives the following counterpart of (4)

$$\begin{aligned}\fint_{Q_{\sigma_{i+1}}} (u-k)_+^{qp^-}\,\mathrm{d}x \le & C\left(\frac{|A(k,\sigma_i)|}{|Q_{\sigma_i}|}\right)^{1-qp^-/(p^-)^*} \\ & \times 2^{iqp^+}\left(\fint_{Q_{\sigma_i}} (u-k)_+^{qp^-}\,\mathrm{d}x + R^{qp^-}\frac{|A(k,\tau)|}{|Q_\tau|}\right)\end{aligned}$$

and the rest of the proof is similar. □

If we take Lemma 6.1 as a starting point in proving the estimates, the assumptions $\int_Q |u|^{p(x)}\,\mathrm{d}x \le 1$ and $\int_Q |\nabla u|^{p(x)}\,\mathrm{d}x \le 1$ are not needed. Indeed, the only restriction on the size of Q in Lemma 6.1 is the requirement that we can find $s > p^+ - p^-$. This restriction depends only on $p(\cdot)$.

Theorem 6.2 *Let u be a quasiminimizer and let Q, R, q and s be as in Lemma 4.7. Then for every $l > 0$ and $\rho < R$, we have the estimate*

$$\operatorname*{ess\,sup}_{Q_\rho} u \le k_0 + R + C\left(\frac{1}{(R-\rho)^n}\int_{Q_R}(u-k_0)_+^l\,dx\right)^{1/l}. \tag{6.3}$$

The constant C depends on l, n, $p(\cdot)$, q, K and the $L^{q's}(Q)$-norm of u.

Proof. We use the same notation as in the proof of Theorem 4.7. With Lemma 6.1, (4.9) becomes

$$\fint_{Q_{\sigma_{i+1}}}(u-k_{i+1})^{qp^-}\,\mathrm{d}x \le \frac{C2^{i\kappa}}{d^{qp^- \varepsilon}}\left(\fint_{Q_{\sigma_i}}(u-k_i)_+^{qp^-}\,\mathrm{d}x\right)^{1+\varepsilon}((R/d)^{qp^-}+1),$$

where $\kappa = q(p^+ + (1+\varepsilon)p^-)$. Recalling that $\Phi_i = d^{-qp^-}\fint_{Q_{\sigma_i}}(u-k_i)_+^{qp^-}\,\mathrm{d}x$, we can write

$$\Phi_{i+1} \le C2^{i\kappa}\Phi_i^{1+\varepsilon}((R/d)^{qp^-}+1).$$

The condition (4.10) now reads

$$\fint_{Q_R}(u-k_0)_+^{qp^-}\,\mathrm{d}x \le \frac{Cd^{qp^-}}{((R/d)^{qp^-}+1)^{1/\varepsilon}}.$$

We may simply choose

$$d = R + C\left(\fint_{Q_R}(u-k_0)_+^{qp^-}\,\mathrm{d}x\right)^{1/(qp^-)},$$

to ensure that $R/d \le 1$. After this, the rest of the proof of Theorem 4.7, and the arguments in Section 4 hold with minor modifications. □

Lemma 6.4 *Let u be a nonnegative quasiminimizer in Ω, and assume that*

$$u \le C/R^\alpha. \tag{6.5}$$

Then there exists a constant $\gamma_0 \in (0,1)$, depending on n, $p(\cdot)$, q, K, α, and the constant of (6.5) C such that if

$$|D(\vartheta, R)| \le \gamma_0 |Q_R|$$

for some $\vartheta > 0$, then

$$\operatorname*{ess\,inf}_{Q_{R/2}} u + R \ge \frac{\vartheta}{2}.$$

Proof. We may assume that $\vartheta \le C/R^\alpha$. Indeed, if $\vartheta > C/R^\alpha$, there is nothing to prove since the condition

$$|D(\vartheta, R)| \le \gamma_0 |Q_R|$$

is never satisfied, since $\gamma_0 < 1$. Next we want to make sure that the constant in (6.3) can be taken to be independent of R. We recall from the proof of Lemma 4.3 that the dependency on the function under consideration can be written as

$$C = \widetilde{C}(1 + \|u\|^{p^+-p^-}_{L^{q's}(Q)}),$$

where $p^+ = \sup_{x\in Q} p(x)$ and $p^- = \inf_{x\in Q} p(x)$. Hence it follows from log-Hölder continuity (2.4) and (6.5) that

$$\|u/R^\alpha\|^{p^+-p^-}_{L^{q's}(Q)} \le C\|u\|^{p^+-p^-}_{L^{q's}(Q)} \le C,$$

with the bound depending only on $p(\cdot)$, α and the constant of (6.5), not on R. After these observations, repeating the proof of Lemma 5.1 gives the claim. $\square$

Lemma 6.6 *Suppose that the hypothesis of Lemma 6.4 holds. For every $\gamma \in (0,1)$ there exists a constant $\mu > 0$, depending on γ, n, $p(\cdot)$, q, K, and the $L^{q's}(Q)$-norm of u, such that if*

$$|D(\vartheta, R)| \le \gamma |Q_R|$$

for some $\vartheta > 0$, then

$$\operatorname*{ess\,inf}_{Q_{R/2}} u + R \ge \mu\vartheta.$$

Proof. Again we may assume that $\vartheta \le C/R^\alpha$. We argue as in Lemma 5.2 up to the inequality (5.3). After this, we employ the harsh estimate

$$|\nabla v|^{p^-} \le (|\nabla v + 1)^{p^-} \le C|\nabla v|^{p(x)} + C,$$

where v is the auxiliary function defined in the proof of Lemma 5.2. We have

$$\int_{D(k,R)} |\nabla v|^{p^-}\,\mathrm{d}x \le \int_{D(k,R)} |\nabla v|^{p(x)}\,\mathrm{d}x + CR^n \le Ck^{p^+}R^{n-p^+} + CR^n \tag{6.7}$$

by (5.4). Since $R < k$, we have $R^n \le k^{p^-}R^{n-p^-}$, and by log-Hölder continuity (2.4) $R^{n-p^+} \le CR^{n-p^-}$. We use the assumption $\vartheta \le C/R^{-\alpha}$ and recall that $k < \vartheta$, so that $k^{p^+} = k^{p^+-p^-}k^{p^-} \le CR^{-\alpha(p^+-p^-)}k^{p^-} \le Ck^{p^-}$, where we again used log-Hölder continuity, and also (6.5). We insert these estimates into (6.7) and obtain

$$\int_{D(k,R)} |\nabla v|^{p^-}\,\mathrm{d}x \le Ck^{p^-}R^{n-p^-}, \tag{6.8}$$

with a constant depending on α and the constant of (6.5). This is the same inequality as (5.5) in the proof of Lemma 5.2. Hence the rest of the proof of Lemma 5.2 holds verbatim. $\square$

An analysis of the proof of Theorem 5.7 shows that the value of i_0 chosen in Lemma 5.2 determines the constant of the weak Harnack inequality. This choice in turn depends on the constant of (5.5). Thus using Lemmas 6.4 and 6.6 to prove the weak Harnack inequality yields a dependence on the constant in (6.5), since (6.5) is used to prove (6.8).

Given a nonnegative quasiminimizer u, we can apply the local boundedness result, and take $C = \sup_{x \in Q_{4R}} u(x)$ in (6.5). We emphasize the fact that the supremum of the *original* quasiminimizer u remains in the estimates, *not* the supremum of u/R^α. Hence Corollary 5.13 holds for u/R^α with a constant depending on the supremum of u, but not on R. We multiply Harnack's inequality

$$\operatorname*{ess\,sup}_{Q(x_0,R)} u/R^\alpha \leq C(\operatorname*{ess\,inf}_{Q(x_0,R)} u/R^\alpha + R),$$

by R^α, and get the following theorem.

Theorem 6.9 *Let u be a nonnegative quasiminimizer in Ω. Then there exists a constant C such that*

$$\operatorname*{ess\,sup}_{Q(x_0,R)} u \leq C(\operatorname*{ess\,inf}_{Q(x_0,R)} u + R^{1+\alpha})$$

for every cube $Q(x_0, R)$ for which $Q(x_0, 10R) \subset \Omega$ and every $\alpha \geq 0$. The constant C depends on n, $p(\cdot)$, q, K, β and the $L^\infty(Q)$ norm of u.

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