VARIATIONAL PARABOLIC CAPACITY

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Abstract. We establish a variational parabolic capacity in a context of degenerate parabolic equations of p-Laplace type, and show that this capacity is equivalent to the standard capacity. As an application, we compute capacities of several explicit sets.

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1. Introduction

Capacity is a central tool in the classical potential theory. It is utilized for example in boundary regularity criterions, characterizations of polar sets and removability results. In the stationary case, capacity has turned out to be the right gauge instead of the Lebesgue measure for exceptional sets for Sobolev functions.

In this work, we study a capacity related to nonlinear parabolic partial differential equations. The principal prototype we have in mind is the p-parabolic equation

$$\partial_t u - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0$$

with $p \geq 2$. In [13], we defined the nonlinear parabolic capacity of a set $E \subset \Omega_\infty = \Omega \times (0, \infty)$ as

$$\operatorname{cap}(E, \Omega_\infty) = \sup\{\mu(\Omega_\infty) : \operatorname{supp} \mu \subset E, 0 \leq u_\mu \leq 1\},$$

where μ is a nonnegative Radon measure, and u_μ is a weak solution to the measure data problem

$$\begin{cases} \partial_t u - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = \mu, & \text{in } \Omega_\infty \\ u(x,t) = 0, & \text{for } (x,t) \in \partial_p \Omega_\infty. \end{cases}$$

The capacity defined this way has many favorable features, including inner and outer regularity, as well as subadditivity to mention a few. The main motivation to study such a capacity is its possible applications to questions regarding boundary regularity and removability. The above capacity is analogous to thermal capacity $p = 2$ related to the heat equation, which together with its generalizations have been studied for example by Lanconelli [20, 21], Watson [29], Evans and Gariepy [7], as well as Gariepy and Ziemer [8, 9]. In the elliptic case, the reader can consult [12].

However, computing the capacities of explicit sets using the above definition is quite challenging. Again, the situation can be compared to the elliptic case, where explicit calculations are usually based on minimizing a variational p-Dirichlet energy integral over a suitable set of test functions. This is the so-called variational capacity. Our objective is to develop tools for estimating capacities of explicit sets in the nonlinear parabolic context. In analogy to the elliptic situation, a central role is played by the nonlinear

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parabolic variational capacity $\operatorname{cap}_{\mathrm{var}}(K,\Omega_\infty)$, see Definitions 4 and 5. Our main result (Theorem 4.7) shows that

$$c^{-1}\operatorname{cap}_{\mathrm{var}}(K,\Omega_\infty) \le \operatorname{cap}(K,\Omega_\infty) \le c^2 \operatorname{cap}_{\mathrm{var}}(K,\Omega_\infty)$$

for $c \equiv c(n,p) > 1$ and for a compact set $K \subset \Omega_\infty$. As an application, in Section 5, we estimate the capacities of space-time curves (Theorem 5.1), cylinders (Theorem 5.3) and hypergraphs (Theorem 5.4). In addition, we give a lower bound for $\operatorname{cap}_{\mathrm{var}}$ in terms of the elliptic capacity in Theorem 5.2.

We first establish the main result in the special case that K is a finite union of space-time cylinders. The simple structure of such sets helps us in deriving estimates using mollified test functions, since we have a better control over the size of the mollified function. As an intermediate step, we prove the equivalence between the energy capacity, defined in (4.1), and the capacity defined above. The proof is based on using the capacitary potential (or balayage/réduite) as a test functions in the measure data problem, and a straightforward estimation.

Then we go on establishing the equivalence between the variational and energy capacities in two main steps: First, in Theorem 4.2, given a nonnegative supersolution u, we construct, by using a backwards in time equation with a right hand side depending on u, a solution v. The suitably chosen exponents in our definitions allow us to bound the key variational quantity $||v||_{\mathcal{W}}$ in terms of the energy of u, $||u||_{\mathrm{en}}$, by a direct estimation utilizing the backwards-in-time equation.

Second, in Theorem 4.3, given v, we show that there exists a supersolution u such that $||u||_{\mathrm{en}} \le c\,||v||_{\mathcal{W}}$ in a suitable intrinsic geometry. Such u is obtained as a solution to the obstacle problem using rescaled v as an obstacle. The above inequality is then derived from the definition of u being a supersolution, in essence using the difference between the rescaled u and v as the test function. This establishes the main result for finite unions of space time cylinders in Ω_T. To complete the proof, we approximate a compact set with unions of cylinders and pass to the limit $T \to \infty$.

Our work owes inspiration and techniques to the work of Pierre [25] for the heat equation, and can be seen as a nonlinear generalization of Pierre's results. For other, but quite different generalizations, see [6], [26] and [27]. Finally, the results in this paper generalize to a wider class of equations of p-parabolic type even if for expository reasons we only work with the p-parabolic equation.

2. Preliminaries

2.1. **Parabolic spaces.** We begin by describing the basic notation. In what follows, $B(x_0,r)=\{x\in\mathbb{R}^n : |z-x|<r\}$ stands for the usual Euclidean ball in $\mathbb{R}^n$. If U' is a bounded subset of an open set U and the closure of U' belongs to U, we write $U' \Subset U$. We denote

$$U_{t_1,t_2} := U\times(t_1,t_2), \qquad \Omega_T := \Omega\times(0,T) \quad \text{and} \quad \Omega_\infty := \Omega\times(0,\infty).$$

Furthermore, the *parabolic boundary* of a cylinder $U_{t_1,t_2} := U\times(t_1,t_2)\subset\mathbb{R}^{n+1}$ is

$$\partial_p U_{t_1,t_2} = (\overline{U}\times\{t_1\}) \cup (\partial U\times(t_1,t_2]).$$

We define the *parabolic boundary* of a finite union of open cylinders $U^i_{t^i_1,t^i_2}$ as

$$\partial_p\Big(\bigcup_i U^i_{t^i_1,t^i_2}\Big) := \Big(\bigcup_i \partial_p U^i_{t^i_1,t^i_2}\Big)\setminus\bigcup_i U^i_{t^i_1,t^i_2}.$$

Note that the parabolic boundary is by definition compact. We let $a \approx b$ denote that there exists a positive constant c depending only on n and p such that $c^{-1}a \leq b \leq ca$.

Next, let U be a bounded open set in $\mathbb{R}^n$. As usual, $W^{1,p}(U)$ denotes the space of real-valued functions f such that $f \in L^p(U)$ and the distributional first partial derivatives $\partial f/\partial x_i$, $i = 1, 2, \ldots, n$, exist in U and belong to $L^p(U)$. We use the norm $\|f\|_{W^{1,p}(U)} = \|f\|_{L^p(U)} + \|\nabla f\|_{L^p(U)}$. The Sobolev space with zero boundary values, $W_0^{1,p}(U)$, is the closure of $C_0^\infty(U)$ with respect to the Sobolev norm. By Sobolev's inequality, we may endow $W_0^{1,p}(U)$ with the norm $\|f\|_{W_0^{1,p}(U)} = \|\nabla f\|_{L^p(U)}$.

By the *parabolic Sobolev space* $L^p(t_1, t_2; W^{1,p}(U))$, with $t_1 < t_2$, we mean the space of measurable functions $u(x,t)$ such that the mapping $t \mapsto u(x,t)$ belongs to $W^{1,p}(U)$ for almost every $t_1 < t < t_2$ and the norm

$$\|u\|_{L^p(t_1,t_2;W^{1,p}(U))} := \left(\int_{t_1}^{t_2} \|u(\cdot,t)\|_{W^{1,p}(U)}^p \, dt\right)^{1/p}$$

is finite. The definition of the space $L^p(t_1, t_2; W_0^{1,p}(U))$ is similar. Analogously, by the space $C(t_1, t_2; L^q(U))$, $t_1 < t_2$ and $q \geq 1$, we mean the space of functions $u(x,t)$, such that the mapping $t \mapsto \int_U |u(x,t)|^q \, dx$ is continuous on the time interval $[t_1, t_2]$.

2.2. **Nonlinear parabolic problems.** We can now introduce the notion of weak solution to

$$\partial_t u - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0. \tag{2.1}$$

Definition 1. A function $u \in L^p_{\mathrm{loc}}(0, T; W^{1,p}_{\mathrm{loc}}(\Omega))$ is a *weak supersolution* to the p-parabolic equation in Ω_T, if

$$\iint_{\Omega_T} \left(|\nabla u|^{p-2} \nabla u \cdot \nabla \phi - u \, \partial_t \phi\right) dx \, dt \geq 0$$

for every $\phi \in C_0^\infty(\Omega_T)$, $\varphi \geq 0$. It is a *weak subsolution*, if the integral above is instead nonpositive. A function u is a *weak solution* in Ω_T if it is both a super- and subsolution in Ω_T, i.e.,

$$\iint_{\Omega_T} \left(|\nabla u|^{p-2} \nabla u \cdot \nabla \phi - u \, \partial_t \phi\right) dx \, dt = 0$$

for every $\phi \in C_0^\infty(\Omega_T)$.

In this work we consider weak solutions with zero boundary data, that is, zero boundary values on the lateral boundary $\partial\Omega \times (0,T)$ and zero initial values on $\overline{\Omega} \times \{t = 0\}$. By this we mean that $u \in L^p(0, T, W_0^{1,p}(\Omega))$ and

$$\lim_{h \to 0} \frac{1}{h} \int_0^h \int_\Omega |u|^2 dz = 0.$$

Similarly, for nonzero boundary data $g \in L^p(0; T; W^{1,p}(\Omega))$, having an L^2 Lebesgue instant at zero, we require

$$\begin{gathered} \lim_{h \to 0} \frac{1}{h} \int_0^h \int_\Omega |u - g|^2 dz = 0 \quad \text{and} \\ u(\cdot,t) - g(\cdot,t) \in L^p(0, T; W_0^{1,p}(\Omega), \quad \text{for almost every } t \in (0,T). \end{gathered} \tag{2.2}$$

In what follows, we often choose a supersolution with zero boundary data and above 1 on a compact set $K \Subset \Omega_T$. In this case, we can always choose our function so that for small enough ε, $u = 0$ in $\Omega \times (0, \varepsilon)$, and thus takes zero initial values in any reasonable sense.

Closely related to weak supersolutions, is the more general class of p-superparabolic functions in $\Theta \subset \mathbb{R}^{n+1}$, see [11].

Definition 2. A function $u : \Theta \to (-\infty, \infty]$ is *p-superparabolic* if

(i) u is lower semicontinuous;
(ii) u is finite in a dense subset of Θ;
(iii) u satisfies the following comparison principle on each space-time box $Q_{t_1,t_2} \Subset \Theta$: If h is p-parabolic in Q_{t_1,t_2} and continuous on $\overline{Q}_{t_1,t_2}$, and if $h \le u$ on $\partial_p Q_{t_1,t_2}$, then $h \le u$ in the whole Q_{t_1,t_2}.

We recall the following theorem from [19].

Theorem 2.1. *Let u be a weak supersolution in Ω_T. Then the lower semicontinuous regularization $\hat{u}$ is a weak supersolution and $u = \hat{u}$ almost everywhere in Ω_T.*

Vice versa we also have the following theorem of [17].

Theorem 2.2. *Let u be a p-superparabolic and locally bounded, then u is a weak supersolution.*

Let u be a supersolution. Then by the Riesz representation theorem, there exists a Radon measure μ_u such that u solves the measure data problem

$$\iint_{\Omega_T} \left(|\nabla u|^{p-2} \nabla u \cdot \nabla \phi - u \, \partial_t \phi \right) \, dx \, dt = \iint_{\Omega_T} \phi \, d\mu_u \tag{2.3}$$

for every $\phi \in C_0^\infty(\Omega_T)$. Conversely, for every finite positive Radon measure, there is a superparabolic function, see for example [4, 15] and [16].

Next we introduce the parabolic obstacle problem, see [2], [18], [23], and also [5]. The following definition for the obstacle problem for $\psi \in C(\overline{\Omega}_T)$ is taken from [23]. In potential theory, the function in the definition below is often called the *balayage*. We denote the lower semicontinuous regularization of u by

$$\hat{u}(x,t) = \liminf_{(y,s) \to (x,t)} u = \lim_{r \to 0} \inf_{B_r(x) \times (t - r^p, t + r^p)} u.$$

Definition 3. Let $\psi \in C(\overline{\Omega}_T)$, and consider the class

$$\mathcal{S}_\psi = \{ u \, : \, u \text{ is essliminf-regularized weak supersolution}, u \ge \psi \text{ in } \Omega_T \}.$$

Define the function

$$w(x,t) = \inf_u u(x,t),$$

where the infimum is taken over the whole class $\mathcal{S}_\psi$. We say that its regularization

$$u(x,t) := \hat{w}(x,t)$$

is the solution to the obstacle problem.

The solution to the obstacle problem has the following basic properties, see again [18] and [23]:

(i) $u \in C(\overline{\Omega}_T)$,
(ii) u is a weak solution in the set $\{(x,t) \in \Omega_T \, : \, u(x,t) > \psi(x,t)\}$, and
(iii) u is the smallest weak supersolution above ψ , i.e. if v is a weak supersolution in Ω_T and $v \ge \psi$, then $v \ge u$.

Continuity of the obstacle can be dropped in the definition of the obstacle problem without losing (iii). Indeed, a special case we are often going to utilize is the characteristic functions of a compact set $K \Subset \Omega_\infty$

$$\psi = \chi_K.$$

We denote the solution to this obstacle problem by $\hat{R}_K$. This function is sometimes called a balayage/réduite, and it can also be seen as a capacitary potential for the following reason: $\hat{R}_K$ is a supersolution by Theorem 2.2, and thus there is a Radon measure μ_K related to this solution through (2.3). Moreover, $\operatorname{supp} \mu_K \subset K$ and it is shown in [13, Theorem 5.7] that

$$\operatorname{cap}(K, \Omega_\infty) = \mu_K(K). \tag{2.4}$$

2.3. **Parabolic capacities.** Next define the functional spaces

$$\mathcal{V}(\Omega_T) = L^p(0, T; W_0^{1,p}(\Omega)), \quad \mathcal{V}'(\Omega_T) = (L^p(0, T; W_0^{1,p}(\Omega)))'$$

with norms

$$\|v\|_{\mathcal{V}(\Omega_T)} = \left(\int_{\Omega_T} |\nabla v|^p \, dx \, dt\right)^{1/p}, \quad \|v\|_{\mathcal{V}'(\Omega_T)} = \sup_{\|\phi\|_{\mathcal{V}(\Omega_T)} \le 1, \phi \in C_0^\infty(\Omega_T)} \left| \int_{\Omega_T} v\phi \, dx \, dt \right|,$$

and also define

$$\mathcal{W}(\Omega_T) = \{u \in \mathcal{V}(\Omega_T) \; : \; \partial_t u \in \mathcal{V}'(\Omega_T)\}$$

equipped with the natural norm $\|u\|_{\mathcal{V}} + \|\partial_t u\|_{\mathcal{V}'}$, which can be equivalently written as

$$\|u\|_{\mathcal{V}(\Omega_T)} + \|\partial_t u\|_{\mathcal{V}'(\Omega_T)} = \|u\|_{\mathcal{V}(\Omega_T)} + \sup_{\|\phi\|_{\mathcal{V}(\Omega_T)} \le 1, \phi \in C_0^\infty(\Omega_T)} \left| \int_{\Omega_T} u \partial_t \phi \, dx \, dt \right|.$$

A first observation, when generalizing the approach in [25] to the nonlinear setting, is that one of the fundamental structures of the p-parabolic equation (2.1) is invariance w.r.t. intrinsic rescaling. Let u be a p-superparabolic function in Ω_∞, then we can define its energy as follows

$$\|u\|_{\mathrm{en}, \Omega_T} = \sup_{0<t<T} \frac{1}{2} \int_\Omega u^2(x, t) \, dx + \int_0^T \int_\Omega |\nabla u|^p \, dx \, dt.$$

If we instead consider $v(x, t) = \lambda^{-1} u(x, \lambda^{2-p} t)$, then v is still p-superparabolic in Ω_∞ and its energy has changed as follows

$$\|v\|_{\mathrm{en}, \Omega_\infty} = \frac{\|u\|_{\mathrm{en}, \Omega_\infty}}{\lambda^2}.$$

We would like $\operatorname{cap}_{\mathrm{var}}$ to reflect this, and therefore define the anisotropic quantity in $\mathcal{W}$ as

$$\|v\|_{\mathcal{W}(\Omega_T)} := \|v\|_{\mathcal{V}(\Omega_T)}^p + \|\partial_t v\|_{\mathcal{V}'(\Omega_T)}^{p'}$$

where $1/p + 1/p' = 1$. The above quantity now scales as λ^2 w.r.t. intrinsic rescaling, but in order to encode the geometry within the definition, we set

Definition 4. For any compact set $K \Subset \Omega_T$, we define

$$\operatorname{cap}_{\mathrm{var}}(K, \Omega_T) = \inf\{\lambda^2 : \; \lambda^2 = \|v\|_{\mathcal{W}(\Omega_{\lambda^{2-p}T})}, \; v \in C_0^\infty(\Omega \times \mathbb{R}), \; v \ge \chi_K\}.$$

If $T = \infty$, we instead set

$$\operatorname{cap}_{\mathrm{var}}(K, \Omega_\infty) = \inf\{\|v\|_{\mathcal{W}(\Omega_\infty)} \; : \; v \in C_0^\infty(\Omega \times \mathbb{R}), \; v \ge \chi_K\}.$$

A couple of remarks are in order. First, note that we have defined it for compact sets. Second, although being intrinsic in nature via the anisotropic nature of $\|\cdot\|_{\mathcal{W}(\Omega_T)}$, the capacity $\mathrm{cap}_{\mathrm{var}}(K,\Omega_\infty)$ only minimizes w.r.t. a quasinorm without any intrinsic conditions. Third, note that for an arbitrary $v \in \mathcal{W}$ we can always find a solution $\lambda \geq 0$ to the equation

$$\lambda^2 = \|v\|_{\mathcal{W}(\Omega_{\lambda^{2-p}T})}.$$

In fact, since λ^2 is strictly increasing and for a given v, $\|v\|_{\mathcal{W}(\Omega_{\lambda^{2-p}T})}$ is nonincreasing we see that for each smooth v there exists a unique solution λ to the above equation. We define the variational capacity for more general sets in a usual way:

Definition 5. Let $U \subset \Omega_T$ be an open set, then we define the intrinsic variational capacity as the limit of exhaustions of compact sets, i.e.

$$\mathrm{cap}_{\mathrm{var}}(U,\Omega_T) = \sup\{\mathrm{cap}_{\mathrm{var}}(K,\Omega_T) \,:\, K \text{ is compact, and } K \subset U\}.$$

For Borel sets B we define it as follows,

$$\mathrm{cap}_{\mathrm{var}}(B,\Omega_T) = \inf\{\mathrm{cap}_{\mathrm{var}}(U,\Omega_T) \,:\, U \text{ is open, and } B \subset U\}.$$

In lack of a better name, we have taken liberty to call the above quantity the variational capacity, due to its connections to the capacity as well as due to the elliptic analogy.

3. Properties of the Variational capacity

We start by listing some very basic properties of the variational capacity. For this, let $\Omega' \subset \Omega$ and $0 < T_1 \leq T \leq T_2 \leq +\infty$. Let K, K_1, and K_2 be compact sets of $\Omega'_T := \Omega' \times (0,T)$ such that $K_1 \subset K_2$. Then the following properties hold:

$$\mathrm{cap}_{\mathrm{var}}(K,\Omega_T) < +\infty$$

$$\mathrm{cap}_{\mathrm{var}}(K_1,\Omega_T) \leq \mathrm{cap}_{\mathrm{var}}(K_2,\Omega_T) \tag{3.1}$$

$$\mathrm{cap}_{\mathrm{var}}(K,\Omega_T) \leq \mathrm{cap}_{\mathrm{var}}(K,\Omega'_T) \tag{3.2}$$

$$\mathrm{cap}_{\mathrm{var}}(K,\Omega_{T_1}) \leq \mathrm{cap}_{\mathrm{var}}(K,\Omega_{T_2}). \tag{3.3}$$

The next lemma turns out the be crucial in what follows. It allows us to reduce the analysis to finite collections of space-time cylinders instead of general compact sets.

Lemma 3.1. *Let K_i, $i = 1, 2, \dots$ be compact sets in Ω_T such that $K_1 \supset K_2 \supset \dots$, then*

$$\lim_{i\to\infty} \mathrm{cap}_{\mathrm{var}}(K_i,\Omega_T) = \mathrm{cap}_{\mathrm{var}}(\cap_i K_i,\Omega_T)\,.$$

Proof. Let $K := \cap_i K_i$. It holds by (3.1) that

$$\mathrm{cap}_{\mathrm{var}}(K,\Omega_T) \leq \mathrm{cap}_{\mathrm{var}}(K_i,\Omega_T)$$

and by passing to a limit $i \to \infty$ ($\mathrm{cap}_{\mathrm{var}}(K_i,\Omega_T)$ is decreasing), we get that $\mathrm{cap}_{\mathrm{var}}(K,\Omega_T) \leq \lim_{i\to\infty} \mathrm{cap}_{\mathrm{var}}(K_i,\Omega_T)$.

To prove the reverse inequality, the idea is to choose $v \geq \chi_K$ which can be used to approximate $\mathrm{cap}_{\mathrm{var}}(K,\Omega_T)$ closely. Then multiplying v by constant slightly larger than 1, we get an admissible test function for the capacity of K_i for i large enough. Yet, as the constant is close to one, we only make a small error.

To work out the details, set $\lambda^2 = \mathrm{cap}_{\mathrm{var}}(K,\Omega_T)$. For $\varepsilon > 0$, there exists $v \in C_0^\infty(\Omega\times\mathbb{R})$, $v \geq \chi_K$ such that $\lambda_v^2 = \|v\|_{\mathcal{W}(\Omega_{\lambda_v^{2-p}T})}$ and

$$\lambda_v^2 \leq \mathrm{cap}_{\mathrm{var}}(K,\Omega_T) + \varepsilon/2.$$

Next, note that since v is smooth we know that for any $\gamma > 0$ there exists $i_0 := i_0(\gamma)$ such that

$$v_\gamma := (1-\gamma)^{-1} v \geq \chi_{K_i}$$

for $i \geq i_0$. Hence for λ_γ satisfying $\lambda_\gamma^2 = \|v_\gamma\|_{\mathcal{W}(\Omega_{\lambda_\gamma^{2-p}T})}$ we have

$$\lambda_\gamma^2 \geq \operatorname{cap}_{\mathrm{var}}(K_i, \Omega_T).$$

Furthermore, by scaling properties

$$\|v_\gamma\|_{\mathcal{W}(\Omega_{\lambda_v^{2-p}T})} \leq (1-\gamma)^{-p} \lambda_v^2.$$

It also holds that

$$\lambda_v^2 \leq \lambda_\gamma^2 \leq (1-\gamma)^{-p} \lambda_v^2.$$

Indeed, the first inequality holds by definition of v_γ, and the second from this and the previous estimate. We now see that

$$\operatorname{cap}_{\mathrm{var}}(K, \Omega_T) \leq \operatorname{cap}_{\mathrm{var}}(K_i, \Omega_T) \leq \lambda_\gamma^2 \leq (1-\gamma)^{-p} \lambda_v^2 \leq (1-\gamma)^{-p} \big(\operatorname{cap}_{\mathrm{var}}(K, \Omega_T) + \varepsilon \big)$$

for any $i \geq i_0(\gamma)$. Letting first $i \to \infty$ and then $\gamma \to 0$, we see that

$$\operatorname{cap}_{\mathrm{var}}(K, \Omega_T) \leq \lim_{i \to \infty} \operatorname{cap}_{\mathrm{var}}(K_i, \Omega_T) \leq \operatorname{cap}_{\mathrm{var}}(K, \Omega_T) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude the proof. $\square$

Lemma 3.2. *Let K be a compact set in Ω_∞. Then*

$$\lim_{T \to \infty} \operatorname{cap}_{\mathrm{var}}(K, \Omega_T) = \operatorname{cap}_{\mathrm{var}}(K, \Omega_\infty)\,.$$

Proof. First, notice that by we have that $T \mapsto \operatorname{cap}_{\mathrm{var}}(K, \Omega_T)$ is an increasing function by (3.3). Set $\lambda_T^2 = \operatorname{cap}_{\mathrm{var}}(K, \Omega_T)$ and since the sequence $(\lambda_T)_T$ is increasing, the limit below exists and we have

$$\lambda^2 := \lim_{T \to \infty} \lambda_T^2 = \lim_{T \to \infty} \operatorname{cap}_{\mathrm{var}}(K, \Omega_T) \leq \operatorname{cap}_{\mathrm{var}}(K, \Omega_\infty) < \infty.$$

From the above we can find large enough T_τ such that by setting $\tau/2 := \lambda^{2-p} T_\tau$ we have $K \subset \Omega_\tau$. Furthermore for any $T > T_\tau$ we can find a $v \in C_0^\infty(\Omega \times \mathbb{R})$ such that $v \geq \chi_K$ and

$$\lambda^2 \geq \lambda_v^2 := \|v\|_{\mathcal{W}(\Omega_{\lambda_v^{2-p}T})} \geq \lambda_T^2 - \varepsilon.$$

Let $\theta \in C_0^\infty(-\infty, 2\tau)$ be such that $\theta = 1$ in $(0,\tau)$, $0 \leq \theta \leq 1$ and $|\theta'| \leq 2/\tau$. We have

$$\begin{aligned}
\big|\langle \partial_t (v\theta), \phi \rangle_{\mathcal{V}(\Omega_\infty)}\big| &= \left| \int_0^{2\tau} \int_\Omega v \theta \partial_t \phi \, dx \, dt \right| \\
&= \left| \int_0^{2\tau} \int_\Omega v \partial_t (\theta \phi) \, dx \, dt - \int_0^{2\tau} \int_\Omega v \phi \theta' \, dx \, dt \right| \\
&\leq \|v\|_{\mathcal{V}'(\Omega_{2\tau})} \|\phi\|_{\mathcal{V}(\Omega_{2\tau})} + \frac{2}{\tau} \|v\phi\|_{L^1(\Omega_{2\tau})} \\
&\leq \left(\|v\|_{\mathcal{V}'(\Omega_{2\tau})} + \frac{c}{\tau} \|v\|_{L^{p'}(\Omega_{2\tau})} \right) \|\phi\|_{\mathcal{V}(\Omega_{2\tau})} \\
&\leq \left(\|v\|_{\mathcal{V}'(\Omega_{2\tau})} + \frac{c}{\tau} \|v\|_{L^{p'}(\Omega_{2\tau})} \right) \|\phi\|_{\mathcal{V}(\Omega_{2\tau})} \\
&\leq \big(\|v\|_{\mathcal{V}'(\Omega_{2\tau})} + c\tau^{-1/(p-1)} \|v\|_{\mathcal{V}(\Omega_{2\tau})} \big) \|\phi\|_{\mathcal{V}(\Omega_{2\tau})}
\end{aligned}$$

Therefore

$$\|v\theta\|_{\mathcal{W}(\Omega_\infty)} \leq \|v\|_{\mathcal{V}(\Omega_{2\tau})}^p + \big(\|v\|_{\mathcal{V}'(\Omega_{2\tau})} + c\tau^{-1/(p-1)} \|v\|_{\mathcal{V}(\Omega_{2\tau})} \big)^{p'}$$

$$\leq (1-\delta)^{-p'} \|v\|_{\mathcal{W}(\Omega_{2\tau})} + c(\delta,p)\tau^{-p'/(p-1)} \|v\|^{p'}_{\mathcal{V}(\Omega_{2\tau})}$$
$$\leq (1-\delta)^{-p'} \lambda^2 + c(\delta,p)\tau^{-p'/(p-1)} \|v\|^{p'}_{\mathcal{V}(\Omega_{\infty})}$$

for all $\delta \in (0,1)$. But now, by the very definition, θv is an admissible test function to test variational capacity over the whole Ω_∞, and we have that

$$\begin{aligned}\operatorname{cap}_{\mathrm{var}}(K,\Omega_\infty) &\leq \|v\theta\|_{\mathcal{W}(\Omega_\infty)} \\ &\leq (1-\delta)^{-p'} \lim_{T\to\infty} \operatorname{cap}_{\mathrm{var}}(K,\Omega_T) + c(\delta,p)\tau^{-p'/(p-1)} \|v\|^{p'}_{\mathcal{V}(\Omega_\infty)},\end{aligned}$$

which holds for all τ sufficiently large. Letting thus $\tau \to \infty$ and then $\delta \to 0$ finishes the proof. □

4. Equivalences of different capacities

In this section, we first prove the main theorem, the equivalence between the capacity and the variational capacity, in the special case that K is a finite union of space-time cylinders. The structure of such a set is much simpler, and this helps us in deriving estimates using mollified test functions, since we can control the change in the mollification, cf. (4.2). We first prove the equivalence between the energy capacity, defined below, and the capacity. Then we go on establishing the equivalence between the energy and variational capacities.

Later, in Theorem 4.6, we extend the result for any compact set by approximating K by a finite unions of cylinders. Finally, we pass to a limit $T \to \infty$.

4.1. **Energy capacity versus capacity.** To prove Theorem 4.5 let us first introduce an intermediate notion of capacity defined in terms of the energy

$$\|u\|_{\mathrm{en},\Omega_T} = \sup_{0<t<T} \frac{1}{2}\int_\Omega u^2(x,t)\,dx + \int_0^T\int_\Omega |\nabla u|^p\,dx\,dt.$$

The energy capacity is defined as

$$\operatorname{cap}_{\mathrm{en}}(K,\Omega_T) = \inf\{\|u\|_{\mathrm{en},\Omega_T} : u \in \mathcal{V}(\Omega_T), u \text{ is } p\text{-superparabolic in } \Omega_T, u \geq \chi_K\}. \quad (4.1)$$

Theorem 4.1. *Let $K \subset \Omega_T$ be a finite family of compact space-time cylinders. Then*

$$\operatorname{cap}_{\mathrm{en}}(K,\Omega_T) \approx \operatorname{cap}(K,\Omega_\infty).$$

Proof. First, we give a rough description of the steps of the proof. Let

$$u_K = \widehat{R}_K$$

be the capacitary potential of K, and μ_K be the corresponding Radon measure in (2.3). By using (2.4) that is $\operatorname{cap}(K,\Omega_\infty) = \mu_K(\Omega_T)$, and estimating the right hand side from below utilizing (2.3) formally with the test function $\phi = u_K$, we are going to establish $\operatorname{cap}_{\mathrm{en}}(K,\Omega_T) \leq 2\operatorname{cap}(K,\Omega_T)$. The reverse inequality follows in a straightforward manner by testing (2.3) with a supersolution u for which $u = 1$ on K.

To work out the details, let $\chi_{h,t} \in C_0^\infty(0,T)$ be a cutoff function in time approximating $\chi_{(0,t)}$. To be more precise, $\chi_{h,t}$ increases pointwise to $\chi_{(0,t)}$ as $h \to 0$ and $\chi_{h,t} = 1$ on $\chi_{h,t} = 1$ on $[h, t-h]$. Fix h. After a standard density argument, $((u_K)_\varepsilon \chi_{h,t})_\varepsilon$ is an admissible test function in (2.3) for small enough ε. Recall that $(\cdot)_\varepsilon$ is the standard mollification only over the time variable.

By Fubini's theorem, we obtain

$$\mu_K(\Omega_T) \geq \int_0^t \int_\Omega ((u_K)_\varepsilon \chi_{h,t})_\varepsilon d\mu_K$$

$$
\begin{aligned}
&= -\int_0^t\int_\Omega (u_K)_\varepsilon \partial_t (u_K)_\varepsilon \chi_{h,t}\,dxds - \int_0^t\int_\Omega (u_K)_\varepsilon^2 \chi_{h,t}'\,dxds \\
&\quad + \int_0^t\int_\Omega |\nabla u_K|^{p-2}\nabla u_K \cdot \nabla((u_K)_\varepsilon \chi_{h,t})_\varepsilon\,dxds.
\end{aligned}
$$

Now

$$
-\int_0^t\int_\Omega (u_K)_\varepsilon \partial_t (u_K)_\varepsilon \chi_{h,t}\,dxds - \int_0^t\int_\Omega (u_K)_\varepsilon^2 \chi_{h,t}'\,dxds \to \frac{1}{2}\int_\Omega u_K^2(x,t)\,dx
$$

and

$$
\int_0^t\int_\Omega (|\nabla u_K|^{p-2}\nabla u_K)_\varepsilon \cdot \nabla (u_K)_\varepsilon \chi_{h,t}\,dxds \to \int_0^t\int_\Omega |\nabla u_K|^p\,dxds
$$

for almost every $t \in (0,T)$ as first $\varepsilon \to 0$ and then $h \to 0$. Hence

$$
\mu_K(\Omega_T) \geq \frac{1}{2}\int_\Omega u_K^2(x,t)\,dx + \int_0^t\int_\Omega |\nabla u_K|^p\ dx\,ds
$$

follows for almost every $t \in (0,T)$. Taking essential supremum over t leads to

$$
2\mu_K(\Omega_T) \geq \sup_{0<t<T} \frac{1}{2}\int_\Omega u_K^2(x,t)\,dx + \int_0^T\int_\Omega |\nabla u_K|^p\ dx\,dt\,.
$$

On the other hand, since K is a finite union of space-time cylinders, we know that for $\varepsilon, h > 0$ small enough,

$$
4^{-1}\chi_K \leq ((u_K)_\varepsilon \chi_{h,T})_\varepsilon \leq 1. \tag{4.2}
$$

Because of (4.2) and passing to the limit as above we can estimate

$$
\begin{aligned}
4^{-1}\mu_K(\Omega_T) &\leq \frac{1}{2}\int_\Omega u_K^2(x,T)\,dx + \int_0^T\int_\Omega |\nabla u_K|^p\ dx\,dt \\
&\leq \sup_t \frac{1}{2}\int_\Omega u_K^2(x,t)\,dx + \int_0^T\int_\Omega |\nabla u_K|^p\ dx\,dt.
\end{aligned}
$$

Therefore, since $\operatorname{cap}(K,\Omega_\infty) = \mu_K(\Omega_T)$, we obtain

$$
2^{-1}\|u_K\|_{\mathrm{en},\Omega_T} \leq \operatorname{cap}(K,\Omega_\infty) \leq 4\|u_K\|_{\mathrm{en},\Omega_T}, \tag{4.3}
$$

and in fact we have proven

$$
\operatorname{cap}_{\mathrm{en}}(K,\Omega_T) \leq 2\operatorname{cap}(K,\Omega_T).
$$

To prove the other direction, for a supersolution u such that $u = 1$ on K, $u(x,0) = 0$, we obtain by the definition of the capacity and (2.3) for u_K that

$$
\begin{aligned}
4^{-1}\operatorname{cap}(K,\Omega_\infty) &\leq \int (u_\varepsilon \phi_h)_\varepsilon\,d\mu_K \\
&= -\int_0^T\int_\Omega (u_K)_\varepsilon \partial_t u_\varepsilon \chi_{h,T}\,dx\,dt - \int_0^T\int_\Omega (u_K)_\varepsilon u_\varepsilon \chi_{h,T}'\,dx\,dt \\
&\quad + \int_0^T\int_\Omega (|\nabla u_K|^{p-2}\nabla u_K)_\varepsilon \cdot \nabla u_\varepsilon \chi_{h,T}\,dx\,dt.
\end{aligned}
$$

Furthermore, first using integration by parts, and then a test function $((u_K)_\varepsilon\chi_{h,T})_\varepsilon$ together with the fact that u is a weak supersolution, we get

$$
-\int_0^T\int_\Omega (u_K)_\varepsilon \partial_t u_\varepsilon \chi_{h,T}\,dx\,dt = \int_0^T\int_\Omega u\,\partial_t((u_K)_\varepsilon \chi_{h,T})_\varepsilon\,dx\,dt
$$

$$\leq \int_0^T \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla((u_K)_\varepsilon \chi_{h,T})_\varepsilon \, dx\, dt.$$

Using the above two displays, first taking the limit $\varepsilon \to 0$ and then $h \to 0$, gives us with the aid of Young's inequality that

$$\begin{aligned} 4^{-1}\operatorname{cap}(K,\Omega_\infty) \leq & \int_\Omega u_K(x,T)u(x,T)\, dx + \int_0^T \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla u_K \, dx\, dt \\ & + \int_0^T \int_\Omega |\nabla u_K|^{p-2}\nabla u_K \cdot \nabla u \, dx\, dt \\ \leq & \delta \|u_K\|_{\mathrm{en},\Omega_T} + c(\delta)\|u\|_{\mathrm{en},\Omega_T}. \end{aligned}$$

Recalling (4.3) and choosing small enough δ, we may absorb the first term on the right hand side into the left. Further, recalling the definition of $\operatorname{cap}_{\mathrm{en}}$, we get

$$\operatorname{cap}(K,\Omega_\infty) \leq c \operatorname{cap}_{\mathrm{en}}(K,\Omega_T). \qquad \square$$

4.2. **Comparing variational and energy capacity.** In Theorem 4.2, given a nonnegative supersolution $u \in \mathcal{V}(\Omega_T) = L^p(0,T;W_0^{1,p}(\Omega))$, we construct, by using a backwards in time equation with a right hand side depending on u, a solution $v \in \mathcal{W}(\Omega_T) = \{v \in \mathcal{V}(\Omega_T) : \partial_t v \in \mathcal{V}'(\Omega_T)\}$ such that by the comparison principle $v \geq u$. The suitably chosen exponents in the definition of $\|\cdot\|_{\mathcal{W}(\Omega_T)}$ allow us to obtain $\|v\|_{\mathcal{W}(\Omega_T)} \leq c\|u\|_{\mathrm{en},\Omega_T}$ by a direct estimation starting from the backwards-in-time equation.

On the other hand, in Theorem 4.3, given a smooth nonnegative v with zero boundary values, we show that there exists a supersolution u such that $u \geq v$ a.e. and $||u||_{\mathrm{en}} \leq c\,||v||_{\mathcal{W}}$ in a suitable intrinsic geometry. In the proof, we construct u as a solution to the obstacle problem using rescaled v as an obstacle, and then derive the above inequality by a using a suitable test function in the weak equation for u.

Finally, combining these results in Theorem 4.4 we end up with

$$\operatorname{cap}_{\mathrm{var}}(K,\Omega_T) \approx \operatorname{cap}_{\mathrm{en}}(K,\Omega_{\lambda^{2-p}T}).$$

As we already know by Theorem 4.1 that $\operatorname{cap}_{\mathrm{var}}(K,\Omega_T) \approx \operatorname{cap}(K,\Omega_\infty)$, we obtain the main result

$$\operatorname{cap}_{\mathrm{var}}(K,\Omega_\infty) \approx \operatorname{cap}(K,\Omega_\infty),$$

by passing to the limit $T \to \infty$.

The proof of the next theorem follows the ideas in [25]. Indeed, the use of a backward-in-time equation is taken from there.

Theorem 4.2. *For each nonnegative supersolution $u \in \mathcal{V}(\Omega_T)$, there exists $v \in \mathcal{W}(\Omega_T)$ such that $v \geq u$ and*

$$\|v\|_{\mathcal{W}(\Omega_T)} \leq c\|u\|_{en,\Omega_T}$$

with $c = c(p)$.

Proof. Let $\tau \in (0,T)$ be such that $u(\cdot,t)$ takes continuously values $u(\cdot,\tau)$ in L^2 as $t \to \tau$. Such instants are of full measure in $(0,T)$ by Lebesgue's differentiation. Solve then the weak solution to the backwards equation

$$\begin{cases} -\partial_t v - \Delta_p v = -2\Delta_p u \\ v(x,\tau) = u(x,\tau) \end{cases} \tag{4.4}$$

in Ω_T with zero lateral boundary values, see (2.2). The right hand side is naturally interpreted as $\int 2\,|\nabla u|^{p-2}\,\nabla u \cdot \nabla \phi\,dz$. The equation has the unique solution $v \in \mathcal{W}(\Omega_\tau)$, because we know that $u \in \mathcal{V}(\Omega_T)$ and hence $\Delta_p u \in \mathcal{V}'(\Omega_\tau)$.

Now choose the mollified test function $\phi = (v_\varepsilon \chi_{h,\tau})_\varepsilon$, where again $\chi_{h,\tau} = 1$ in $[h, \tau - h]$ and $\chi_{h,\tau} \in C_0^\infty(0,T)$. Testing then the weak formulation of the backwards equation and passing to a limit $\varepsilon \to 0$ similarly as in Theorem 4.1, we get

$$\frac{1}{2}\int_{\Omega_\tau} v^2 \chi'_{h,\tau}\,dx\,dt + \int_{\Omega_\tau} |\nabla v|^p \chi_{h,\tau}\,dx\,dt = 2\int_{\Omega_\tau} |\nabla u|^{p-2}\nabla u \cdot \nabla v \chi_{h,\tau}\,dx\,dt.$$

Passing to the limit as $h \to 0$ we obtain by Young's inequality that

$$-\frac{1}{2}\int_\Omega v^2(x,\tau)\,dx + \frac{1}{2}\int_\Omega v^2(x,0)\,dx + c\int_{\Omega_\tau} |\nabla v|^p\,dx\,dt \le c\int_{\Omega_\tau} |\nabla u|^p\,dx\,dt,$$

which gives together with the terminal data of v that

$$\int_{\Omega_\tau} |\nabla v|^p\,dx\,dt \le c\left(\frac{1}{2}\int_\Omega u^2(x,\tau)\,dx + \int_{\Omega_\tau} |\nabla u|^p\,dx\,dt\right) \le c\|u\|_{\mathrm{en},\Omega_T}. \tag{4.5}$$

Let us now consider the dual norm. We have by (4.4) and Hölder's inequality that

$$\begin{aligned}
\|\partial_t v\|_{\mathcal{V}'(\Omega_\tau)} &= \sup_{\|\phi\|_{\mathcal{V}(\Omega_\tau)} \le 1} \left|\int_{\Omega_\tau} v\,\partial_t \phi\,dx\,dt\right| \\
&\le \sup_{\|\phi\|_{\mathcal{V}(\Omega_\tau)} \le 1} \left[\left|\int_{\Omega_\tau} |\nabla v|^{p-2}\nabla v \cdot \nabla \phi\,dx\,dt\right| + 2\left|\int_{\Omega_\tau} |\nabla u|^{p-2}\nabla u \cdot \nabla \phi\,dx\,dt\right|\right] \\
&\le c\left(\|v\|^p_{\mathcal{V}(\Omega_T)} + \|u\|^p_{\mathcal{V}(\Omega_T)}\right)^{1/p'},
\end{aligned}$$

where $\phi \in C_0^\infty(\Omega_\tau)$ and $1/p + 1/p' = 1$ so that $1/p' = (p-1)/p$. Since

$$\|v\|^p_{\mathcal{V}(\Omega_T)} + \|u\|^p_{\mathcal{V}(\Omega_T)} \le c\|u\|_{\mathrm{en},\Omega_T}$$

holds by (4.5) and definition of $\|u\|_{\mathrm{en},\Omega_T}$, we also get

$$\|\partial_t v\|^{p'}_{\mathcal{V}'(\Omega_T)} \le c\|u\|_{\mathrm{en},\Omega_T}.$$

Hence we conclude that

$$\|v\|_{\mathcal{W}(\Omega_T)} \le c\|u\|_{\mathrm{en},\Omega_T}.$$

To check that $v \ge u$, we do the following formal computation

$$-\partial_t v - \Delta_p v = -2\Delta_p u \ge -\partial_t u - \Delta_p u\,,$$

based on (4.4) and the definition of a supersolution for u. Now we can use the comparison principle for backwards equations to conclude the inequality in Ω_τ. The rigorous treatment goes via weak formulation and standard mollification argument. Indeed, subtracting the backwards equations in the weak form, passing to limits, and using the initial condition, we get for a.e. $s \in (0,\tau)$ that

$$\begin{aligned}
&\int_\Omega (u-v)_+^2(x,s)\,dx - 0 \\
&\qquad \le -\int_{\Omega_\tau} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v) \cdot \nabla(u-v)_+\,dx\,dt \\
&\qquad \le 0.
\end{aligned}$$

This implies that $u \le v$ a.e. in Ω_τ. Finally, as this reasoning holds for almost every $\tau \in (0,T)$, we obtain the statement. □

The proof of Theorem 4.3 utilizes the following rescaling lemma.

Lemma 4.1. *Let $v \in \mathcal{W}(\Omega_\infty)$, and suppose that $\lambda > 0$ is the intrinsic parameter satisfying*

$$\lambda^2 = \|v\|_{\mathcal{W}(\Omega_{\lambda^{2-p}T})}.$$

Further, let

$$\tilde{v}(x,\tau) = \lambda^{-1} v(x, \lambda^{2-p}\tau).$$

Then

$$\|\tilde{v}\|^p_{\mathcal{V}(\Omega_T)} + \|\partial_\tau \tilde{v}\|^{p'}_{\mathcal{V}'(\Omega_T)} = 1. \tag{4.6}$$

Proof. By the definition of the norm $\|\cdot\|_{\mathcal{V}(\Omega_{\lambda^{2-p}T})}$ and $\|\cdot\|_{\mathcal{V}(\Omega_T)}$ we have by changing the variables $t = \lambda^{2-p}\tau$ that

$$\begin{aligned}
\|v\|_{\mathcal{V}(\Omega_{\lambda^{2-p}T})} &= \left(\int_0^{\lambda^{2-p}T}\int_\Omega |\nabla v(x,t)|^p \, dx \, dt\right)^{1/p} \\
&= \left(\int_0^T \int_\Omega |\lambda \nabla \tilde{v}(x,\tau)|^p \, dx \lambda^{2-p} \, d\tau\right)^{1/p} = \lambda^{2/p}\|\tilde{v}\|_{\mathcal{V}(\Omega_T)}.
\end{aligned}$$

To find out the scaling of the norm of the time derivative in the dual space we denote $\bar{\phi}(x,\tau) = \phi(x, \lambda^{2-p}\tau)$ and observe by a similar calculation as above

$$\|\phi\|_{\mathcal{V}(\Omega_{\lambda^{2-p}T})} = \lambda^{(2-p)/p}\|\bar{\phi}\|_{\mathcal{V}(\Omega_T)}.$$

Now

$$\begin{aligned}
\|\partial_t v\|_{\mathcal{V}'(\Omega_{\lambda^{2-p}T})} &= \sup_{\|\phi\|_{\mathcal{V}(\Omega_{\lambda^{2-p}T})} \le 1} |\langle \partial_t v, \phi\rangle| \\
&= \sup_{\|\phi\|_{\mathcal{V}(\Omega_{\lambda^{2-p}T})} \le 1} \left| \int_0^{\lambda^{2-p}T} \int_\Omega v \partial_t \phi \, dx \, dt \right| \\
&= \sup_{\|\phi\|_{\mathcal{V}(\Omega_{\lambda^{2-p}T})} \le 1} \left| \int_0^{\lambda^{2-p}T} \int_\Omega \lambda \tilde{v}(x, \lambda^{p-2}t) \lambda^{p-2} \partial_\tau \bar{\phi}(x, \lambda^{p-2}t) \, dx \, dt \right| \\
&= \sup_{\|\lambda^{(2-p)/p}\bar{\phi}\|_{\mathcal{V}(\Omega_T)} \le 1} \left| \int_0^T \int_\Omega \lambda \tilde{v}(x,\tau) \lambda^{p-2} \partial_\tau \bar{\phi}(x,\tau) \, dx \lambda^{2-p} \, d\tau \right| \\
&= \sup_{\|\lambda^{(2-p)/p}\bar{\phi}\|_{\mathcal{V}(\Omega_T)} \le 1} \left| \lambda^{1+(p-2)/p} \int_0^T \int_\Omega \tilde{v}(x,\tau) \lambda^{(2-p)/p} \partial_\tau \bar{\phi}(x,\tau) \, dx \, d\tau \right| \\
&= \lambda^{2/p'} \sup_{\|\hat{\phi}\|_{\mathcal{V}(\Omega_T)} \le 1} \left| \int_0^T \int_\Omega \tilde{v}(x,\tau) \partial_\tau \hat{\phi}(x,\tau) \, dx \, d\tau \right| \\
&= \lambda^{2/p'} \|\partial_t \tilde{v}\|_{\mathcal{V}'(\Omega_T)},
\end{aligned}$$

because $1 + (p-2)/p = 2/p'$. Here we have also denoted $\hat{\phi} = \lambda^{(2-p)/p}\bar{\phi}$. Therefore

$$\begin{aligned}
\lambda^2 &= \|v\|_{\mathcal{W}(\Omega_{\lambda^{2-p}T})} \\
&= \|v\|^p_{\mathcal{V}(\Omega_{\lambda^{2-p}T})} + \|\partial_t v\|^{p'}_{\mathcal{V}'(\Omega_{\lambda^{2-p}T})} \\
&= \lambda^2 \|\tilde{v}\|^p_{\mathcal{V}(\Omega_T)} + \lambda^2 \|\partial_t \tilde{v}\|^{p'}_{\mathcal{V}'(\Omega_T)} \\
&= \lambda^2 \|\tilde{v}\|_{\mathcal{W}(\Omega_T)}
\end{aligned}$$

holds, which is exactly (4.6) since $\lambda > 0$. $\square$

Theorem 4.3. *Let $v \in C_0^\infty(\Omega \times \mathbb{R})$ be nonnegative. Let λ be the nonnegative number such that*

$$\lambda^2 = \|v\|_{\mathcal{W}(\Omega_{\lambda^{2-p}T})}.$$

Then there exists a continuous nonnegative supersolution u in $\Omega_{\lambda^{2-p}T}$ such that $u \geq v$ and

$$\|u\|_{en,\Omega_{\lambda^{2-p}T}} \leq c\|v\|_{\mathcal{W}(\Omega_{\lambda^{2-p}T})}$$

for a constant $c = c(n,p)$.

Proof. Assume, without loss of generality, that $\lambda > 0$. Indeed, otherwise v is identically zero and we may simply take $u = 0$.

Let $\tilde{v}$ be defined as in Lemma 4.1, then consider the obstacle problem with $\tilde{v}$ as the obstacle in Ω_T. Let $\tilde{u}$ be the continuous solution to this problem. It holds that $\tilde{u}$ is a supersolution and

$$\tilde{u} \geq \tilde{v} \quad \text{in} \quad \Omega_T.$$

Moreover, since $\partial\Omega$ is regular and $\tilde{v}$ is continuous up to the parabolic boundary, $\tilde{u}$ is continuous up to the parabolic boundary as well and $\tilde{u} = \tilde{v}$ on $\partial_p\Omega_T$. Thus, for each $\delta > 0$ we find $\varepsilon > 0$ such that

$$\psi = (((\tilde{u} - \tilde{v} - \delta)_\varepsilon)_+ \chi_{h,\tau})_\varepsilon$$

vanishes on the parabolic boundary. Here $\chi_{h,\tau}$ is again a smooth approximation of a characteristic functions $\chi_{(0,\tau)}$ where $\tau \in (0,T)$, and the subscript ε refers to the standard time mollification. We may use ψ as a nonnegative smooth test function in the weak formulation for $\tilde{u}$. Then using integration by parts, we obtain

$$\begin{aligned}
&\int_0^\tau \int_\Omega \frac{\partial \tilde{u}_\varepsilon}{\partial t} ((\tilde{u} - \tilde{v} - \delta)_\varepsilon)_+ \chi_{h,\tau} \, dx \, dt \\
&+ \int_0^\tau \int_\Omega (|\nabla \tilde{u}|^{p-2} \nabla \tilde{u})_\varepsilon \cdot \nabla ((\tilde{u} - \tilde{v} - \delta)_\varepsilon)_+ \chi_{h,\tau} \, dx \, dt = \int_0^\tau \int_\Omega \psi d\mu_{\tilde{u}}.
\end{aligned} \tag{4.7}$$

From this we obtain

$$\begin{aligned}
\int_0^\tau \int_\Omega & \frac{\partial \tilde{u}_\varepsilon}{\partial t} ((\tilde{u} - \tilde{v} - \delta)_\varepsilon)_+ \chi_{h,\tau} \, dx \, dt \\
&= \frac{1}{2} \int_0^\tau \int_\Omega \frac{\partial ((\tilde{u} - \tilde{v} - \delta)_\varepsilon)_+^2}{\partial t} \chi_{h,\tau} \, dx \, dt \\
&\quad + \int_0^\tau \int_\Omega \frac{\partial \tilde{v}_\varepsilon}{\partial t} ((\tilde{u} - \tilde{v} - \delta)_\varepsilon)_+ \chi_{h,\tau} \, dx \, dt \,.
\end{aligned} \tag{4.8}$$

Now for the first term on the right hand side, by the definition of $\mathcal{W}$-space and the properties of standard mollifiers, we obtain that

$$\int_0^\tau \int_\Omega \frac{\partial \tilde{v}_\varepsilon}{\partial t} ((\tilde{u} - \tilde{v} - \delta)_\varepsilon)_+ \chi_{h,\tau} \, dx \, dt \geq -\|\tilde{v}\|_{\mathcal{W}(\Omega_T)} \|\tilde{u} - \tilde{v}\|_{\mathcal{V}(\Omega_\tau)} \geq -\|\tilde{u} - \tilde{v}\|_{\mathcal{V}(\Omega_\tau)} \,.$$

Here we also used Lemma 4.1. For the second term on the right hand side of (4.8), integration by parts, on the other hand, gives

$$\frac{1}{2} \int_0^\tau \int_\Omega \frac{\partial ((\tilde{u} - \tilde{v} - \delta)_\varepsilon)_+^2}{\partial t} \chi_{h,\tau} \, dx \, dt = -\frac{1}{2} \int_0^\tau \int_\Omega ((\tilde{u} - \tilde{v} - \delta)_\varepsilon)_+^2 \chi_{h,\tau}'(t) \, dx \, dt \,.$$

Combining the previous two displays in (4.8), passing to a limit first in ε and then in δ, and using Lemma 4.1, we get

$$\limsup_{\delta,\varepsilon \to 0} \int_0^\tau \int_\Omega \frac{\partial \tilde{u}_\varepsilon}{\partial t} ((\tilde{u} - \tilde{v} - \delta)_\varepsilon)_+ \chi_{h,\tau} \, dx \, dt$$

$$\geq -\|\tilde{u}\|_{\mathcal{V}(\Omega_\tau)} - 1 - \frac{1}{2}\int_0^\tau \int_\Omega (\tilde{u}-\tilde{v})^2 \chi'_{h,\tau}(t)\,dx\,dt.$$

Next, by Young's inequality and Lemma 4.1, we get

$$\begin{aligned}
\lim_{\delta,\varepsilon\to 0} \int_0^\tau \int_\Omega & (|\nabla \tilde{u}|^{p-2}\nabla \tilde{u})_\varepsilon \cdot \nabla((\tilde{u}-\tilde{v}-\delta)_\varepsilon)_+ \chi_{h,\tau}\,dx\,dt \\
&= \int_0^\tau \int_\Omega (|\nabla \tilde{u}|^{p-2}\nabla \tilde{u}) \cdot \nabla(\tilde{u}-\tilde{v})\chi_{h,\tau}\,dx\,dt \\
&\geq \int_0^\tau \int_\Omega |\nabla \tilde{u}|^p \chi_{h,\tau}\,dx\,dt - \int_0^\tau \int_\Omega |\nabla \tilde{u}|^{p-1}|\nabla \tilde{v}|\chi_{h,\tau}\,dx\,dt \\
&\geq \frac{1}{p}\int_0^\tau \int_\Omega |\nabla \tilde{u}|^p \chi_{h,\tau}\,dx\,dt - \frac{1}{p}\int_0^\tau \int_\Omega |\nabla \tilde{v}|^p \chi_{h,\tau}\,dx\,dt \\
&\geq \frac{1}{p}\int_0^\tau \int_\Omega |\nabla \tilde{u}|^p \chi_{h,\tau}\,dx\,dt - \frac{1}{p}.
\end{aligned}$$

Finally, recall that since the obstacle $\tilde{v}$ is continuous, the solution $\tilde{u}$ is continuous and hence $\psi_{\delta,\varepsilon} \to 0$ on $\{\tilde{u} = \tilde{v}\}$ uniformly as $\delta, \varepsilon \to 0$. In addition, by the properties of the obstacle problem $\operatorname{supp} \mu_{\tilde{u}} \subset \{\tilde{u} = \tilde{v}\}$. Thus combining the previous estimates with (4.7), we conclude that

$$\frac{1}{p}\int_0^\tau \int_\Omega |\nabla \tilde{u}|^p \chi_{h,\tau}\,dx\,dt - \frac{1}{2}\int_0^\tau \int_\Omega (\tilde{u}-\tilde{v})^2 \chi'_{h,\tau}(t)\,dx\,dt \leq \|\tilde{u}\|_{\mathcal{V}(\Omega_\tau)} + \frac{p+1}{p}\,.$$

Passing to a limit $h \to 0$, using the initial condition, and choosing τ to be a Lebesgue instant such that $\int_\Omega \tilde{u}^2(x,\tau)\,dx \geq \frac{1}{2}\sup_{0<t<T}\int_\Omega \tilde{u}^2\,dx$, we end up with

$$\int_0^T \int_\Omega |\nabla \tilde{u}|^p\,dx\,dt + \sup_{0<t<T}\int_\Omega \tilde{u}^2\,dx \leq c.$$

To get rid of the term $\int_\Omega \tilde{v}^2\,dx\big|_0^\tau$ on the left hand side, we used the fact that $C(0,T;L^2(\Omega)) \hookrightarrow \mathcal{W}(\Omega_T)$, together with Lemma 4.1. Now by changing variables $u(x,t) = \lambda\tilde{u}(x,\lambda^{p-2}t)$ we obtain the estimate

$$\|u\|_{\mathrm{en},\Omega_{\lambda^{2-p}T}} \leq c\lambda^2 = c\|v\|_{\mathcal{W}(\Omega_{\lambda^{2-p}T})},$$

where u is a supersolution satisfying $u \geq v$. This finishes the proof. □

Next we combine the previous two theorems to obtain $\operatorname{cap}_{\mathrm{var}}(K,\Omega_T) \approx \operatorname{cap}_{\mathrm{en}}(K,\Omega_{\lambda^{2-p}T})$. When combining the previous results, we would like to take v to be smooth. The following lemma gives us the appropriate mollification. By following the proof of [6, Appendix A], together with the fact, that on a finite union of space-time cylinders, we can control the space-time mollification, cf. (4.2). In addition, close to the lateral boundary of Ω_T, the mollification can be done by using the partition of unity, as usual. The details are left for the reader.

Lemma 4.2. *Let K be a compact set consisting of a finite union of space-time cylinders. Let $v \in \mathcal{W}(\Omega_T)$ be such that $v \geq \chi_K$ for a compact set $K \Subset \Omega_T$. Then there is $w \in C_0^\infty(\Omega\times\mathbb{R})$ such that $w \geq \chi_K$ and*

$$\|w\|_{\mathcal{W}(\Omega_T)} \leq c\|v\|_{\mathcal{W}(\Omega_T)}.$$

Theorem 4.4. *Let K be a compact set consisting of a finite union of compact space-time cylinders, set $\lambda^2 = \operatorname{cap}_{\mathrm{var}}(K,\Omega_T)$, and suppose that $K \Subset \Omega_{\lambda^{2-p}T}$. Then*

$$\operatorname{cap}_{\mathrm{var}}(K,\Omega_T) \approx \operatorname{cap}_{\mathrm{en}}(K,\Omega_{\lambda^{2-p}T}),$$

where $\Omega_{\lambda^{2-p}T}$ is interpreted as Ω_∞ if $\lambda = 0$.

Proof. Suppose first that $\operatorname{cap}_{\mathrm{var}}(K,\Omega_T) > 0$. To compare the variational and energy capacities, we first define

$$\lambda^2 = \operatorname{cap}_{\mathrm{var}}(K,\Omega_T).$$

Given $\delta > 0$, choose a superparabolic function $u \in \mathcal{V}(\Omega_{\lambda^{2-p}T})$ such that $u \geq \chi_K$ and

$$\|u\|_{\mathrm{en},\Omega_{\lambda^{2-p}T}} \leq \operatorname{cap}_{\mathrm{en}}(K,\Omega_{\lambda^{2-p}T}) + \delta.$$

Using Theorem 4.2 we find a function $v \in \mathcal{W}(\Omega_{\lambda^{2-p}T})$ such that $v \geq u \geq \chi_K$ and

$$\|v\|_{\mathcal{W}(\Omega_{\lambda^{2-p}T})} \leq c\|u\|_{\mathrm{en},\Omega_{\lambda^{2-p}T}}.$$

By Lemma 4.2, we may take v to be smooth. Furthermore, associated to v there is $\lambda_v \geq \lambda$ such that $\lambda_v^2 = \|v\|_{\mathcal{W}(\Omega_{\lambda_v^{2-p}T})}$ and

$$\operatorname{cap}_{\mathrm{var}}(K,\Omega_T) \leq \lambda_v^2 = \|v\|_{\mathcal{W}(\Omega_{\lambda_v^{2-p}T})} \leq c\|u\|_{\mathrm{en},\Omega_{\lambda^{2-p}T}} \leq c(\operatorname{cap}_{\mathrm{en}}(K,\Omega_{\lambda^{2-p}T}) + \delta).$$

This gives

$$\operatorname{cap}_{\mathrm{var}}(K,\Omega_T) \leq c \operatorname{cap}_{\mathrm{en}}(K,\Omega_{\lambda^{2-p}T}).$$

Conversely, for small enough $\delta > 0$ let us now consider $v \in C_0^\infty(\Omega\times\mathbb{R}) \cap \mathcal{W}(\Omega_{\lambda^{2-p}T})$ such that $v \geq \chi_K$ and

$$\lambda^2 \leq \|v\|_{\mathcal{W}(\Omega_{\lambda_v^{2-p}T})} = \lambda_v^2 \leq (1+\delta)\lambda^2,$$

which we find by the definition of $\operatorname{cap}_{\mathrm{var}}$. Theorem 4.3 yields a superparabolic function $u \in \mathcal{V}(\Omega_{\lambda_v^{2-p}T})$ such that $u \geq v$ and

$$\|u\|_{\mathrm{en},\Omega_{\lambda_v^{2-p}T}} \leq c\|v\|_{\mathcal{W}(\Omega_{\lambda^{2-p}T})} \leq c(\operatorname{cap}_{\mathrm{var}}(K,\Omega_T) + \delta).$$

Since K is compact and belongs to $\Omega_{\lambda^{2-p}T}$, we find small enough δ so that $K \Subset \Omega_{\lambda_v^{2-p}T}$ as well. Extending u to the entire cylinder Ω_∞ as a solution with initial values at the time $t_0 = \lambda_v^{2-p}T$ equal to u, we see that

$$\|u\|_{\mathrm{en},\Omega_{\lambda^{2-p}T}} \leq \|u\|_{\mathrm{en},\Omega_\infty} \leq 2\|u\|_{\mathrm{en},\Omega_{\lambda_v^{2-p}T}} \leq c(\operatorname{cap}_{\mathrm{var}}(K,\Omega_T) + \delta\lambda^2), \tag{4.9}$$

where the second inequality is due to an energy inequality

$$\int_{t_0}^\infty \int_\Omega |\nabla u|^p \, dx\, dt + \sup_{t_0<t} \int_\Omega u^2(x,t)\, dx \leq c \int_\Omega u^2(x,t_0)\, dx$$

for a solution.

The estimate (4.9) immediately gives that

$$\operatorname{cap}_{\mathrm{en}}(K,\Omega_{\lambda^{2-p}T}) \leq c \operatorname{cap}_{\mathrm{var}}(K,\Omega_T),$$

concluding the proof when $\lambda > 0$.

To finish the proof, we consider the case $\operatorname{cap}_{\mathrm{var}}(K,\Omega_T) = 0$. For any $\delta > 0$ there exists $v \in C_0^\infty(\Omega\times\mathbb{R}) \cap \mathcal{W}(\Omega_\infty)$, $v \geq \chi_K$, such that

$$\delta > \lambda_v^2 = \|v\|_{\mathcal{W}(\Omega_{\lambda_v^{2-p}T})}.$$

For this given v, we may argue as in the first step using Theorem 4.3. Indeed, we find u such that $u \geq v \geq \chi_K$ and

$$\|u\|_{\mathrm{en},\Omega_\infty} \leq c\|v\|_{\mathcal{W}(\Omega_{\lambda_v^{2-p}T})} < c\delta^2.$$

Therefore $\operatorname{cap}_{\mathrm{en}}(K,\Omega_\infty) = 0$ and the proof is finished in all cases. $\square$

4.3. **Comparing the capacity and the variational capacity.**

Theorem 4.5. *Let $K \subset \Omega_T$ be a compact set consisting of a finite union of compact space-time cylinders $K = \bigcup_i \overline{Q}^i_{t_1^i, t_2^i}$, and let $\lambda^2 = \operatorname{cap}_{\mathrm{var}}(K, \Omega_T)$. If $K \Subset \Omega_{\lambda^{2-p}T}$, then*

$$\operatorname{cap}_{\mathrm{var}}(K, \Omega_T) \approx \operatorname{cap}(K, \Omega_\infty),$$

where $\Omega_{\lambda^{2-p}T}$ is interpreted as Ω_∞ if $\lambda = 0$.

Proof. The proof immediately follows from Theorem 4.1 and Theorem 4.4. $\square$

Now we ready to prove the main result. We start with a local version.

Theorem 4.6. *Let $K \Subset \Omega_T$ be a compact set and assume that for $\lambda^2 = \operatorname{cap}_{\mathrm{var}}(K, \Omega_T)$ we have $K \Subset \Omega_{\lambda^{2-p}T}$. Then*

$$\operatorname{cap}_{\mathrm{var}}(K, \Omega_T) \approx \operatorname{cap}(K, \Omega_\infty),$$

where $\Omega_{\lambda^{2-p}T}$ is interpreted as Ω_∞ if $\lambda = 0$.

Proof. Let $\{K_i\}_{i=1}^\infty$ be a nested sequence of compact sets, each a finite union of space-time cylinders, such that

$$\bigcap_{i=1}^\infty K_i = K.$$

Then from Lemma 3.1 we see that

$$\lim_{i \to \infty} \operatorname{cap}_{\mathrm{var}}(K_i, \Omega_T) = \operatorname{cap}_{\mathrm{var}}(K, \Omega_T) = \lambda^2. \tag{4.10}$$

First there exists an i_1 such that if $i \geq i_1$, $K_i \Subset \Omega_T$. Second since λ_i is a decreasing sequence, we get that there is an i_2 such that $K_i \Subset \Omega_{\lambda^{2-p}T} \subset \Omega_{\lambda_i^{2-p}T}$ holds for all $i \geq i_2$.

Now, for $i \geq \max\{i_1, i_2\}$, Theorem 4.5 gives

$$\operatorname{cap}_{\mathrm{var}}(K_i, \Omega_T) \approx \operatorname{cap}(K_i, \Omega_\infty).$$

Employing the outer regularity of $\operatorname{cap}(\cdot, \Omega_\infty)$ (see [13, Lemma 5.8]) together with (4.10) completes the proof. $\square$

Our main theorem now immediately follows.

Theorem 4.7. *Let K be a compact set of Ω_∞. Then*

$$\operatorname{cap}_{\mathrm{var}}(K, \Omega_\infty) \approx \operatorname{cap}(K, \Omega_\infty).$$

Proof. Combine Theorem 4.6 with Lemma 3.2. $\square$

5. Estimates of capacities for explicit sets

In this section we prove specific estimates for capacity. First let us define standard elliptic capacity for a compact set as

$$\operatorname{cap}_{\mathrm{e}}(K, \Omega) = \inf \left\{ \int_\Omega |\nabla u|^p \, dx \, : \, u \geq \chi_K, \, u \in C_0^\infty(\Omega) \right\}.$$

Theorem 5.1. *Let $K \subset \Omega$ be a compact set such that $\operatorname{cap}_{\mathrm{e}}(K, \Omega) = 0$. Let $\phi : [t_1, t_2] \to \Omega$, $0 < t_1 < t_2 < T$, be a Lipschitz continuous function and let the set K_ϕ be defined as*

$$K_\phi := \{(x + \phi(t), t) \, : \, x \in K, t \in [t_1, t_2]\}.$$

Then

$$\operatorname{cap}_{\mathrm{var}}(K_\phi, \Omega_T) = 0.$$

Proof. Let also $K_\varepsilon = \{x : d(x,K) < \varepsilon\}$ for $\varepsilon > 0$. Then also the closure of $U := \{(x+\phi(t),t) : x \in K_\varepsilon, t \in [t_1,t_2]\}$ belongs to $\Omega \times \mathbb{R}$ and it covers K_ϕ if $\varepsilon > 0$ is small enough. By the assumptions we find a smooth function $u \in C_0^\infty(K_\varepsilon)$ such that

$$\Big(\int_{K_\varepsilon} |\nabla u|^p \, dx\Big)^{1/p} < \varepsilon^2. \tag{5.1}$$

Let us now consider the function

$$v(x,t) := u(\phi(t)+x)\theta(t),$$

where $\theta \in C_0^\infty(-\varepsilon/2+t_1, t_2+\varepsilon/2)$, $\theta = 1$ on $[t_1,t_2]$ as well as $|\theta'| \le 2/\varepsilon$, and we also define $\phi(t) := \phi(t_1)$ when $t < t_1$ as well as $\phi(t) := \phi(t_2)$ when $t > t_2$. Then $v \in W_0^{1,\infty}(\Omega \times \mathbb{R})$ and $v \ge \chi_{K_\phi}$. Strictly speaking this is not an admissible smooth test function since ϕ is only Lipschitz, but this point could easily be overcome by an approximation argument.

From (5.1) we get that

$$\|v\|^p_{\mathcal{V}(\Omega_\infty)} \le c(t_2 - t_1 + \varepsilon)\varepsilon^{2p}.$$

We also see that

$$\partial_t v(x,t) = \partial_t \phi \cdot \nabla u(\phi(t)+x)\theta(t) + u(\phi(t)+x)\theta'(t),$$

and consequently

$$|\partial_t v(x,t)| \le \|\partial_t \phi\|_\infty |\nabla u(\phi(t)+x)| + |\theta'(t)|\,|u(\phi(t)+x)|\,.$$

Thus we get

$$\begin{aligned}\|\partial_t v\|_{L^{p'}(\Omega\times(-\varepsilon/2+t_1,\varepsilon/2+t_2))} &\le c(t_2-t_1+\varepsilon)^{1/p'}\|\partial_t\phi\|_\infty\|\nabla u\|_{L^{p'}(K_\varepsilon)} + c\varepsilon^{-1}\|u\|_{L^{p'}(K_\varepsilon)} \\ &\le c(t_2-t_1+\varepsilon)^{1/p'}\|\partial_t\phi\|_\infty\varepsilon^2 + c\varepsilon\end{aligned}$$

where we also utilized Sobolev's inequality $\|\nabla u\|_{L^{p'}(K_\varepsilon)} \le c\|\nabla u\|_{L^p(K_\varepsilon)} \le c\varepsilon^2$ and $\|u\|_{L^{p'}(K_\varepsilon)} \le c\|\nabla u\|_{L^p(K_\varepsilon)} \le c\varepsilon^2$. Thus, for suitably small $\varepsilon > 0$, we obtain

$$\|\partial_t v\|_{\mathcal{V}'(\Omega_\infty)} \le c_1\varepsilon.$$

for a constant $c_1 = c_1(t_2 - t_1, \|\partial_t\phi\|_\infty, |\Omega|, n, p) > 1$. Letting ε to zero finishes the proof. □

A point has a zero elliptic p-capacity if and only if $p \le n$, see for example Section 2.11 [10]. From this and the previous lemma we have the following corollary.

Corollary 5.1. *Let $\phi : [t_1,t_2] \to \Omega$ be a Lipschitz continuous function with $0 < t_1 < t_2 < T$ and define $\Phi = \{(\phi(t),t) : t \in [t_1,t_2]\}$. Then*

$$\operatorname{cap}_{\mathrm{var}}(\Phi,\Omega_T) = 0$$

if and only if $2 \le p \le n$.

Next, we will derive a lower bound for the variational capacity in terms of the elliptic capacity. Since we are going to consider time slices, we need the following convenient notational tool, the t-slice of $E \subset \mathbb{R}^{n+1}$ is defined as follows

$$\pi_t(E) = \{x : (x,t) \in E\} \subset \mathbb{R}^n.$$

Theorem 5.2. *Let $K \Subset \Omega_T$ be a compact set and let $\lambda^2 = \operatorname{cap}_{\mathrm{var}}(K,\Omega_T)$. Then*

$$\int_0^{\lambda^{2-p}T} \operatorname{cap}_{\mathrm{e}}(\pi_t(K),\Omega)\,dt \le \operatorname{cap}_{\mathrm{var}}(K,\Omega_T).$$

Proof. Let $v \in C_0^\infty(\Omega \times \mathbb{R})$ be such that $\lambda_v^2 = \|v\|_{\mathcal{W}(\Omega_{\lambda_v^{2-p}T})} < \lambda^2 + \varepsilon$. Then

$$\operatorname{cap_e}(\pi_t(K), \Omega) \leq \int_\Omega |\nabla v(x,t)|^p \, dx,$$

and hence

$$\int_0^{\lambda_v^{2-p}T} \operatorname{cap_e}(\pi_t(K), \Omega) \, dt \leq \int_0^{\lambda_v^{2-p}T} \int_\Omega |\nabla v(x,t)|^p \, dx \, dt \leq \|v\|_{\mathcal{W}(\Omega_{\lambda_v^{2-p}T})} < \lambda^2 + \varepsilon$$

follows. Letting ε to zero then finishes the proof. □

Lemma 5.1. *Let $1 < p < n$ and $Q_r = B(0,r) \times (t_0 - \tau, t_0)$ such that $Q_r \Subset \Omega_T$. Let $\lambda^2 = \operatorname{cap_{var}}(Q_r, \Omega_T)$. If $Q_r \subset \Omega_{\lambda^{2-p}T}$, then*

$$\operatorname{cap_{var}}(\overline{Q}_r, \Omega_T) \geq c^{-1} \tau r^{n-p}$$

with $c = c(n,p)$.

Proof. We know that

$$\operatorname{cap_e}(B(0,r), \Omega) \geq \operatorname{cap_e}(B(0,r), \mathbb{R}^n) \geq c^{-1} r^{n-p},$$

see for example [1, 10, 24]. Using Theorem 5.2 we conclude that

$$\operatorname{cap_{var}}(\overline{Q}_r, \Omega_T) \geq c^{-1} \tau r^{n-p}.$$ □

The converse holds as well.

Lemma 5.2. *Let $1 < p < n$, $Q_r = B(0,r) \times (t_0 - \tau, t_0)$, $\Omega = B(0, 2r)$ and assume that $Q_r \Subset \Omega_T$. Then there exists a constant $c = c(n,p)$ such that*

$$\operatorname{cap}(\overline{Q_r}, \Omega_\infty) \leq c(r^n + \tau r^{n-p}).$$

Proof. Let u solve

$$\begin{cases} -\triangle_p u = 0, & \text{in } \Omega \setminus B(0,r) \\ u = 1, & \text{on } \overline{B}(0,r) \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Then

$$\int_\Omega |\nabla u|^p \, dx \approx r^{n-p}. \tag{5.2}$$

Furthermore, $0 \leq u \leq 1$ and u is a supersolution to the p-Laplace equation in Ω. Next, define the function

$$v(x,t) := \begin{cases} 0, & \text{if } (x,t) \in \Omega \times (-\infty, t_0 - \tau) \\ u(x), & \text{if } (x,t) \in \Omega \times [t_0 - \tau, t_0) \\ h(x,t), & \text{if } (x,t) \in \Omega \times [t_0, +\infty), \end{cases}$$

where $h(x,t)$ is the solution to the Dirichlet problem

$$\begin{cases} h_t - \Delta_p h = 0, & \text{in } \Omega \times (t_0, \infty) \\ h(\cdot, t) = 0, & \text{on } \partial\Omega \times [t_0, \infty) \\ h(\cdot, t) = u(\cdot), & \text{in } \Omega \times \{t_0\}. \end{cases}$$

Then $v(x,t)$ is a supersolution in Ω_∞ satisfying $v \geq \chi_{\overline{Q_r}}$. To see this, it suffices, since v is bounded, to observe that v satisfies a comparison principle, cf. for example Lemma 2.9

in [3], Theorem 2.2 or Theorem 1.1 in [14]. Moreover, since h is a solution in $\Omega \times (t_0, \infty)$, we have the usual energy estimate

$$\sup_{t>t_0} \frac{1}{2} \int_\Omega h^2(x,t)\, dx + \int_{t_0}^\infty \int_\Omega |\nabla h|^p \, dx\, dt \le \frac{1}{2} \int_\Omega u^2(x, t_0)\, dx \le \frac{1}{2} \int_\Omega 1 \, dx \le c r^n .$$

Combining this together with (5.2), and using Theorem 4.1 we see that

$$\operatorname{cap}(\overline{Q}_r, \Omega_\infty) \le c \|v\|_{\mathrm{en},\Omega_\infty} \le c(r^n + \tau r^{n-p}). \qquad \square$$

Theorem 5.3. *Let $Q_r = B(0, r) \times (t_0 - r^p, t_0)$, and assume that $Q_{2r} \subset \Omega_T$. Then*

$$\operatorname{cap}(\overline{Q_r}, \Omega_\infty) \approx r^n.$$

Proof. Follows from (3.2), Lemma 5.2, Lemma 5.1 and Theorem 4.7. $\square$

Let us now state a useful comparison lemma. Observe that earlier, we only worked the equivalence between the capacity and the energy capacity for a finite union of cylinders whereas the lemma below is for any compact set.

Lemma 5.3. *Let $K \subset \Omega_\infty$ be a compact set. Then there exists a constant $c = c(n, p) > 1$ such that*

$$\operatorname{cap}_{\mathrm{en}}(K, \Omega_\infty) \le c \operatorname{cap}(K, \Omega_\infty).$$

Proof. There is a shrinking sequence of compact sets $K_i \subset \Omega_\infty$, $i = 1, 2, \dots$, consisting of finite unions of space time cylinders such that $\cap_i K_i = K$. Since $\operatorname{cap}_{\mathrm{en}}(\cdot, \Omega_\infty)$ is an increasing set function, the conclusion of the lemma follows easily from [13, Lemma 5.8]. $\square$

Our next theorem is in some sense a parabolic counterpart to the fact that the elliptic capacity only "sees" the external boundary, i.e.

$$\operatorname{cap}_{\mathrm{e}}(K, \mathbb{R}^n) = \operatorname{cap}_{\mathrm{e}}(\partial_e K, \mathbb{R}^n),$$

where $\partial_e K$ is the *external boundary*, that is, the boundary of the unbounded component of the complement of K. See for example [22] or [28].

Theorem 5.4. *Let $Q_r^+ = B(x_0, r) \times (t_0, t_0 + \tau)$ be such that $Q_{2r} \subset \Omega_\infty$ and let*

$$\mathcal{H} = \{(x, h(x)) : x \in \overline{B}(x_0, r)\}$$

where $h \in C(\mathbb{R}^n)$ satisfies $h(x) = t_0$ on $\partial B(x_0, r)$ and $\mathcal{H} \subset Q_r^+$. Then

$$c^{-1} \left(\int_0^\infty \operatorname{cap}_{\mathrm{e}}(\pi_t(\mathcal{H}), \Omega)\, dt + r^n \right) \le \operatorname{cap}(\mathcal{H}, \Omega_\infty) \le c(r^n + \tau\, r^{n-p})$$

with $c = c(n, p)$.

Proof. The bound from above follows immediately from Lemma 5.2. Let us consider the lower bound. To this end, for any $\varepsilon > 0$ we find v such that $\|v\|_{\mathrm{en},\Omega_\infty} \le \operatorname{cap}_{\mathrm{en}}(\mathcal{H}, \Omega_\infty) + \varepsilon$. Since v is p-superparabolic and $v \ge \chi_{\mathcal{H}}$, we have by the lower semicontinuity that $1 \le v(z) \le \liminf_{y \to z} v(y)$ whenever $z \in \mathcal{H}$. Define then

$$\widetilde{\mathcal{H}} := \{(x, t) \,:\, x \in B(x_0, r)\,, t \in (t_0, h(x))\},$$

i.e., the set of all the space-time points lying between the graphs (x, t_0) and $(x, h(x))$. Set now

$$\tilde{v}(x,t) = \begin{cases} \min(1, v(x,t)), & \text{if } (x,t) \notin \widetilde{\mathcal{H}} \\ 1, & \text{if } (x,t) \in \widetilde{\mathcal{H}}. \end{cases}$$

Note that $\tilde{v}$ is lower semicontinuous in $\Omega \times (t_0, \infty)$ and hence it is p-superparabolic in $\Omega \times (t_0, \infty)$ by the pasting lemma in [3].

Let us now consider two cases,

(A) $|\pi_{t_0}(\mathcal{H})| \geq 1/2|B(x_0, r)|$,

(B) Alternative (A) does not hold.

In alternative (A), we know that $v \geq 1$ on $\mathcal{H} \cap B(x_0, r) \times \{t_0\}$ and we have a bound for the measure of this set. Next, since v is a bounded p-superparabolic function in Ω_∞, it is also a supersolution by [17, Theorem 5.8]. As such we can see that by testing formally with $v\chi_{\{t>t_0\}}$

$$\frac{1}{2}|B(x_0, r)| \leq \int_\Omega v^2(x, t_0)\, dx \leq 2 \int_{\Omega \times (t_0, \infty)} |\nabla v|^p \, dx\, dt \leq 2\|v\|_{\text{en},\Omega_\infty}, \tag{5.3}$$

where rigorous treatment goes via mollifications.

In the case of alternative (B), we know by the continuity of h that there exists $\sigma > 0$ such that $|\pi_{t_0+\sigma}(\widetilde{\mathcal{H}})| \geq \frac{1}{4}|B(x_0, r)|$, moreover we know that $\tilde{v} \geq 1$ on $\pi_{t_0+\sigma}(\widetilde{\mathcal{H}})$. Again since $\tilde{v}$ is a bounded p-superparabolic function, we can as in (5.3), test formally with $u\chi_{\{t>t_0+\sigma\}}(t)$, and get

$$\begin{aligned}\frac{1}{4}|B(x_0, r)| &\leq \int_\Omega \tilde{v}^2(x, t_0 + \sigma)\, dx \leq 2 \int_{\Omega \times (t_0+\sigma, \infty)} |\nabla \tilde{v}|^p \, dx\, dt \\ &= 2 \int_{\Omega \times (t_0+\sigma,\infty) \setminus \widetilde{\mathcal{H}}} |\nabla v|^p \, dx\, dt \leq 2\|v\|_{\text{en},\Omega_\infty}.\end{aligned}$$

Thus we obtain that in both alternatives (A) and (B) we have $|B(x_0, r)|/4 \leq \|v\|_{\text{en},\Omega_\infty} \leq c\,\text{cap}(\mathcal{H}, \Omega_\infty)$, where in the last inequality we have used Lemma 5.3. Together with Theorem 5.2 we get the desired lower bound by summing up. □

From the above Theorem we can obtain a symmetric upper and lower bound on the capacity of a cylinder, which tells us that the parabolic capacity of a cylinder is essentially the sum of the elliptic capacity of the lateral part integrated and the parabolic capacity of the bottom disc.

Corollary 5.2. *Let $Q_r = B(0, r) \times (t_0 - \tau, t_0)$, such that $Q_{2r} \subset \Omega_\infty$. Then*

$$\text{cap}_{\text{var}}(\overline{Q}_r, \Omega_\infty) \approx r^n + \tau r^{n-p}.$$

Proof. From Lemma 5.1, and Lemma 5.2 we get that

$$\frac{\tau r^{n-p}}{c} \leq \text{cap}_{\text{var}}(\overline{Q}_r, \Omega_T) \leq c(r^n + \tau r^{n-p}).$$

To improve the lower bound, note that $\overline{B(0, r)} \times \{t_0\} \subset \overline{Q}_r$ hence Theorem 5.4, and Theorem 4.6 gives

$$\text{cap}_{\text{var}}(\overline{Q}_r, \Omega_T) \geq \frac{r^n}{c}.$$

□

Knowing that the hypergraph bends up a bit allows us to get a symmetric upper and lower bound even in this case.

Corollary 5.3. *Let $Q_r^+(\tau) = B(0, r) \times (t_0, t_0 + \tau)$ be such that $Q_{2r}^+(\tau) \subset \Omega_\infty$ and let $\mathcal{H}$ be as above. Suppose furthermore that $\mathcal{H} \subset (x_0, t_0) + (\overline{Q}_r^+(\tau) \setminus \overline{Q}_{r/M}^+(\tau/M))$ for some $M > 1$. Then*

$$c^{-1}\left(r^n + \tau\, r^{n-p}\right) \leq \text{cap}(\mathcal{H}, \Omega_T) \leq c\left(r^n + \tau\, r^{n-p}\right)$$

for $c = c(n, p, M)$.

Proof. The upper bound follows from Lemma 5.2. The lower bound, on the other hand, is a consequence of the fact that

$$\mathrm{cap_e}(\pi_t(\mathcal{H}), \Omega) \geq r^n/c,$$

and thus Theorem 5.4 yields the result. □

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