Exercise help set 6 **Topological Vector Spaces**

6.2. *E:n weak topology $\sigma(E,F)$ is Hausdorff exactly if the duality separeates E " Why?*

Solution: In a locally convex space **Hausdorff is equivalent to, that at each point $x \in E \setminus \{0\}$ at lest one seminorm $p_k \in \mathcal{P}$ is $\neq 0$.**

Let $\sigma(E,F)$ be Hausdorff and $x \in E \setminus \{0\}$. Now exists distinct $\sigma(E,F)$- neighbourhoods $U \in \mathcal{U}_0$ and $V \in \mathcal{U}_x$. In particular there are $y_1, \dots y_n \in F$ for which $U \supset \{x \in E \mid |\langle \xi, y_m \rangle| \le 1 \forall m \in \{1, \dots, n\}\}$. Since $x \notin U$, then some $|\langle x, y_m \rangle| > 1 > 0$, so the duality separates E . In a locally convex space Hausdorff is equivalent to , thatat each $x \in E \setminus \{0\}$ at lest one seminorm $p_k \in \mathcal{P}$ is $\neq 0$—evidently.

In revrse, assume that the duality separates . Let $x \in E \setminus \{0\}$. by (!) 7.6. the duality $\langle \cdot, \cdot \rangle$ separates the space E, if $\mathrm{Ker}(F \to E' : x \mapsto \langle \cdot, y \rangle) = \{0\}$, so $x \notin \mathrm{Ker}(F \to E' : x \mapsto \langle \cdot, y \rangle)$, so there exists $y \in F$, s. th. $\langle x, y \rangle \neq 0$. So $\sigma(E,F)$ is Hausdorff.

6.3. *Prove, that if E is locally convex Hausdorff-space, then $\sigma(E,E^*)$ is Hausdorff-topology.*

Solution: This follows fronm ex 1 and theorem 7.8., jby which the topological dual of a Hausdorff-space separates it (by Hahn-Banach).

6.4. *Call a* **locally convex** *topology τ on E* compatible with the duality *(engl: is compatible with) (E,F), if*

$$E^*_\tau = F.$$

Ex, if E is locally convex Hausdorff-space, then weak topology $\sigma(E,E^*)$ is compatible with the duality (E,E^*), , Also evidently E:n original topology. Is $\sigma(E,E^*)$ the finest –or maybe the coarsest (E,E^*)-compatible topology?

Coarsest. $\sigma = \sigma(E,F)$ is locally convex and $E^*_\sigma = F$. No compatible lc top is coarser, since a locally convex topology τ sis compatible exactly when every $|\langle \cdot, y \rangle| \quad (y \in F)$ is τ–continuous .

6.5. *Let (E,F) be a separable duality .*

a) Prove, that a convexlla set A has the same closure in every (E,F)–compatible topology.

Solution: .

By Mazurin/Banach: if A is convex subset , then A is closed, if and only if A is the intersection of some (necessarily closed) hyperplanes in E: (and these are the same in all compatible toppologies!) j

6.6. Let E and F be vector spaces $\dim E < \infty$. find a necessary and sufficient condition for F, which guaantees the existence of a separable duality (E, F).

If separable duality (E, F), then both $x \mapsto \langle x, \cdot \rangle : E \to F'$ and $y \mapsto \langle \cdot, y \rangle : F \to E'$ are injective, so $\dim E \leq \dim F' = \dim F$ and $\dim E = \dim E' \geq \dim F$, siis $\dim E = \dim F$. This was necessary -a nd is also sufficient – do it! .

6.7. *Let E hae the topology $\sigma(E, E')$. Prove, that if $A \subset E$ is bounded, then*
a) exists finite dimensional subspace $G \subset E$ such, that $A \subset G$
b) In E every vector subspace is closed
c) In E every vector subspace has a topological supplement

Solution: $E = E_\sigma$.The topology $\sigma = \sigma(E, E')$ is defined by the seminlorms $p(s) = |f(x)|$, where $f \in E'$.

a) "$A \subset E$ is bounded" means, that every linear form $f \in E'$ is bounded in the set A. Antiteesi exists lin independent vektorit $x_1, x_2 \dots \in A$. Let us define a lin form in the lin subspace $H = \text{span}(x_1, x_2 \dots) = \left\{ \sum_{\mathbf{N}} \lambda_i x_i \;\middle|\; \text{only finitely manyi } \lambda_i \neq 0 \right\}$ by $f(\sum_{\mathbf{N}} \lambda_i x_i) = \sum_{\mathbf{N}} i\lambda_i$, so in particular $f(x_i) = i$ for all $i \in \mathbf{N}$, and A is not σ–bounded.

b) Let $H \subset E$ be a lineaarinen subspace. Let $K_H \subset H$ be a Hameli basis for H and continue it to become a Hamel–basis i K of the whole space. Let us define for each $x \in K \setminus K_H$ a linear form f_x by defining for the basis vextors $f_x(x) = 1$ and $f_x(y) = 0$ for all $x \in K \setminus \{x\}$. Now f is continuous (every linear form is continuous in this topology!), so $\operatorname{Ker} f_x$ is closed and $H = \bigcap_{x \in K \setminus K_H} \operatorname{Ker} f_x$ is closed.

c) Every subspace ahas an algebraic supplement. Since in this topology every subspace is closed, every suoplementti is topologinen.

6.8. *Let E ääretönulotteinen locally convex Hausdorff-space. Prove, that E^*_σ ei ole normiutuva.*

In the weak topology, every neighbourhood of the origin contains a non-zeo vektorisubspace , a finit e intersectionof hyperplanes!. !

6.9. *Let E ääretönulotteinen normispace. Prove, that duaalin nollavektori $0 \in E^*$ kuuluu duaalin yksikköpall is kuoren $\{x* \mid \|x^*\| = 1\}$ sulkeumaan topologyssa $\sigma(E^*, E)$.*

Let $U \in \mathcal{U}$ be a basis neighbourhood, $U = \{x \mid |\langle x, y_n \rangle| \leq 1, n = 1, \dots, n\}$ Now $U \supset \bigcup_{j=1}^{n} \operatorname{Ker} y_j$. the kernels have codimension 1, so in infinte dim space their intersection is infinite codimensional subspace, which evidently intrsects the sphere. .