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Exercise help set 5 **Topological Vector Spaces**

Part I Theory.

5.2. Assume $(E, \mathcal{T}_E)$ and $(F, \mathcal{T}_F)$ are Fréchet spaces and in the space F there also is another Hausdorff–topology τ_F, coarser than $\mathcal{T}_F$. Assume $T : E \to F$ linear.

Prove that if T is continuous $\mathcal{T}_E \to \tau_F$, then it is continuous $\mathcal{T}_E \to \mathcal{T}_F$ (Look at the graph!).

Check that T satisfies the conditions of the closed graph theorem. At least $T : (E, \mathcal{T}_E) \to (F, \mathcal{T}_F)$, where both are Fréchet spaces. Since T is continuous as a mapping $T : (E, \mathcal{T}_E) \to (F, \tau_F)$, its graph is closed (Hausdorff!) in teh product topology $\mathcal{T}_E \times \tau_F$, jwhich is by assumption coarser than $\mathcal{T}_E \times \mathcal{T}_F$. So the graph is is closed in both. $\square$

5.3. *Assume $(E, \mathcal{T}_E)$ and $(F, \mathcal{T}_F)$ Fréchet spaces and assume $(X, \mathcal{T}_X)$ is a Hausdorff-space . Assume $T : E \to F$ linear. Prove that T is continuous, if there exists a continuous injection $f : F \to G$, such that $f \circ T$ is continuous. (Idea: proj topology)*

Write $\tau_F = f^{-1}(\mathcal{T}_X)$ for the projective topology defined by the mapping $f : F \to X$. Since T is injection and X is Hausdorff, also τ_F isHausdorff, (If $x \neq y \in F$, then $f(x) \neq f(y) \in X$, so there exist disjoint $U \in \mathcal{U}_{f(x)}$ and $y \in \mathcal{U}_{f(y)}$ and we find disjoint neighbourhoods $f^{-1}(U) \in \mathcal{U}_x$ and $f^{-1}(V) \in \mathcal{U}_y$.) By assumption f is continuous, so τ_F is coarser than $\mathcal{T}_F$. By assumption also $f \circ T$ is continuous.Thereforei T is continuous as $T : (E, \mathcal{T}_E) \to (F, \tau_F)$: in projective topology a set is open, if and only if its image is open. So: $A \in \tau_F \implies f(A) \in \mathcal{T}_X \implies T^{-1}(A) = T^{-1}(f^{-1}(f(A))) = (f \circ T)^{-1}(f(A)) \in \mathcal{T}_E$. Notice, at $A = f^{-1}(f(A)$ we used the ingormation that A is injective. $\square$

Part II Function spaces and solution to an older problem.

5.4. *Assume $k \in \mathbf{N} \cup \{\infty\}$. In the space $E = \mathcal{C}^k = \mathcal{C}^k(\mathbf{R}) = \{f : \mathbf{R} \to \mathbf{R} \mid f$ is k times differentiable$\}$ the standard topology, also called the topology of* compact $\mathcal{C}^k$–convergence *is the lokaalikonveksi topology, given by the seminorms*

$$\left| \left(\frac{\partial}{\partial x} \right)^{\alpha} f(x) \right| = \sup_{x \in K} |f^{(\alpha)}(x)|.$$

Prove that every $\mathcal{C}^k$ is metrisable and Hausdorff.

Since every compact set is included in some compact interval $[-m, m]$, the seminorm family $\mathcal{P}$ can be replaced by a basis of continuous seminorms: $\mathcal{P}_J = \{p_{\alpha,m} \mid \alpha \in \{0, 1, \ldots, k\}$ and $K = [-m, m] \subset \mathbf{R}$, giving the same topology and being countable. So $\mathcal{C}^k$ is metrizable since it also is Hausdorff: for all $f \in \mathcal{C}^{(k)} \setminus \{0\}$there is some point $x \in \mathbf{R}$, where $f(x) \neq 0$, so $p_{0,m}(f) = \sup_{-m \leq y \leq m} |f^{(0)}(y)| \geq |f(x)| > 0$, for $-m \leq x \leq m$.

5.5. *Prove that every $\mathcal{C}^k$ $(k \in \mathbf{N})$ is (seq)complete , so it is Fréchet. (Banach? Is there a continuousa norm ?)*

Let f_n be a Cauchy sequence in $\mathcal{C}^k$. At each point $x \in \mathbf{R}$ $f : n(x)$ is a Cauchy-sequence in $\mathbf{K}$ so it converges. This defines $f : \mathbf{R} \to \mathbf{R}$. Let $K = [-m, m]$. On K the functions f_n converge uniformly, sillä the seminorm $p_{0,m}$ is in this set the sup-norm giving uniform convergence .For the same reason, the derivatives converge uniformly. By a theorem from analysis, (not immensely difficult to prove), the derivatives converge to f'. Similarly, the second derivatives converge to f'' uniformly in $]-m, m[$ etc by induction all the way to the a kth derivative. Since this works for all m, then $p_{m,\alpha}(f_n - f) \to 0$ for all m in question and α,same as $f_N \to f \in \mathbf{C}^k$ in the correct topology.

5.6. Prove that $\mathcal{C}^\infty$ is (seq)complete , so it is Fréchet. (Banach? Is there a continuousa norm ?)

The same idea works again!!

5.7. Assume $K \subset \mathbf{R}$ compact and $k \in \mathbf{N} \cup \infty$. (One K fixed.) Prove that alis paces $\mathcal{C}^k(K) = \{f \in \mathcal{C}^k \mid \operatorname{supp} f \subset K\}$ are (jono)complete, so Fréchet. (Banach? Is there a continuousa norm ?)

The same idea works again since $f_n \to f$ and $f_n(x) = 0$ for all $x \notin K$.

5.8. Prove that the completion (same as closure!) of $\mathcal{C}^k(K)$ in $\mathcal{C}^k(K)$ is is $\mathcal{C}^\infty(K)$. This means that $\mathcal{C}^\infty(K)$ is dense in $\mathcal{C}^k(K)$ (which is complete).

The well known proof from analysis courses is based on convolutions.

Remark. In analysis, often n-dimensional versions of the spaces $\mathcal{C}^k$ are used: $E = \mathcal{C}^k = \mathcal{C}^k(\mathbf{R}^n) = \{f : \mathbf{R}^n \to \mathbf{R} \mid f \text{ is } k \text{ times diff}\}$ and topology byseminorms

$$p_\alpha(f) = \sup_{x \in K} \left| \left(\frac{\partial}{\partial x} \right)^\alpha f(x) \right|,$$

where $K \subset \mathbf{R}^n$ is compact and $\left(\frac{\partial}{\partial x}\right)^\alpha f(x) = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f(x)$ is the partial derivative correspinds to multi-indeksiä $\alpha = (\alpha_q, \ldots, \alpha_n) \in \mathbf{N}^n$ (so $(\alpha_1 + \cdots + \alpha_n = \alpha)$. More generally, the set $\mathbf{R}^n$ can be replaced by any open set $\Omega \subset \mathbf{R}^N$. All this brings no essential change to what was done above. .

5.9. *Prove that the standard topology of $\mathcal{C}^k$ is also defined by the seminorms $\mathcal{Q}$, of seminorms*

$$q_K(f) = \int_K |f^{(n)}(x)|\, dx,$$

where $K \subset \mathbf{R}$ is compact and $n \in \mathbf{N}$. Hint: $f(x) = \int_x^{x+1} ((t - x - 1)f'(t) + f(t))\, dt$. The hint is okay, since by partial integrtion

$$\int_x^{x+1} (t-x-1)f'(t)\, dt + \int_x^{x+1} f\, dt = /_x^{x+1}\, (t-x-1)f(t) - \int_x^{x+1} f\, dt + \int_x^{x+1} f\, dt = f(x).$$

The standard topology in $\mathcal{C}^\infty$:comes from the seminorms $\mathcal{P} = \{p_{n,K} \mid K \subset \mathbf{R}$ compakt, $n \in \mathbf{N}\}$, where $p_{n,K}(f) = \sup\{|f^{(n)}(x)| \mid x \in K\}$. Call the topologies τ_P and τ_Q.

a) $\tau_Q \subset \tau_P$:

$$q_{n,K}(f) = \int_K |f^{(n)}(x)| \leq \sup_K |f^{(n)}(x)| \int_K 1 = |K| p_{n,K}(f)$$

b) $\tau_P \subset \tau_Q$: Partial integration (or just use the hint) $m < k$

$$f^{(m)}(x) = \int_x^{x+1} (t-x-1) f^{(m+1)}(t) + f^{(m)}(t)\, dt,$$

joten for all $m \in \mathbf{N}$

$$\begin{aligned} p_{m,K}(f) &= \sup_K |f^{(m)}(x)| \\ &= \sup_K \{ | \int_x^{x+1} (t-x-1) f^{(m+1)}(t) + f^{(m)}(t)\, dt | \\ &\leq \sup_K \left(| \int_x^{x+1} (t-x-1) f^{(m+1)}(t)\, dt | + | \int_x^{x+1} f^{(m)}(t)\, dt | \right) \\ &\leq \sup_K \{ | \int_x^{x+1} (t-x-1) f^{(m+1)}(t)\, dt | + \sup_K | \int_x^{x+1} f^{(m)}(t)\, dt | \\ &\leq \sup_{x,y \in K} |y-x| \int_{K'} |f^{(m+1)}(t)|\, dt + \| \int_{K'} f^{(m)}(t)|\, dt \\ &\leq \operatorname{diam} K' \int_{K'} q_{m+1,K'}(f) + q_{m,K'}(f), \end{aligned}$$

where $K' = \overline{\bigcup_{x \in K} B(x,1)}$.

Combination. .

5.10. *Assume $K \subset \mathbf{R}^n$ compact set and $F \subset \mathbf{R}^K$ Banach space , whose elements called vectors, or points) are funstions $K \to \mathbf{R}$ ie $E\pounds isavectorsubspaceof \mathbf{R}^K$. Assume , that the topology in F is finer than the topology of pointwise convergence, which is the product topology in / from $\mathbf{R}^K y$. Assume that $\mathcal{C}^\infty(K) \subset F$.*

We prove , that there exists la number $k \in \mathbf{N}$, such that $C^k(K) \subset F$.

a) Apply ex ä 5.1. choosing: for E: the space C^∞ vwith its standard toplogy now called $\mathcal{T}_E$. For F we choose the norm topology $\mathcal{T}_F = \mathcal{T}_{\|\cdot\|}$ is inclusion $x \mapsto x$.

Prove that the inclusion mapping T is continuous $\mathcal{T}_E \to \mathcal{T}_F$ same as a mapping $C^\infty(K) \to (F, \mathcal{T}_{\|\cdot\|}$.

b) find out, that there exists a number $\lambda > 0$ and a (semi)norm $p_{n,K}$, such that $\| \cdot \|_F \leq \lambda p_{n,K} = \lambda \| \cdot \|_n$. Check (or rememeber) , that the continuous seminormi $f \mapsto p_{n,K}(f) = \sup_{x \in K} |f^{(n)}(x)|$ is in fact a norm in $E = C^\infty(K)$. So in $E = C^\infty(K)$ we have $\mathcal{T}_F \subset \mathcal{T}_{p_{n,K}}$ and all in all

The subspace topology from F $l \subset$ topology from $\mathcal{C}^k_K \subset$ original topology in $\mathcal{C}^\infty_K$

So. explain why

The completion of $\mathcal{C}^k(K) = (\mathcal{C}^\infty(K)$: in $\mathcal{C}^k(K)$:ssa) $\subset$ the completion of $\mathcal{C}^\infty(K)$ in theoriginal norm of F. Are we done? □

tCecking the conditions: $\mathcal{C}^\infty(K)$ and the Banach-space F are Fréchet'n spaces and in F there is also another Hausdorff–topology τ_F, coarser than $\mathcal{T}_F$. (Totea!) The maping $T : E \to F : x \to x$ is linear.

Let us prove that the inclusion mapping T is continuous $C^\infty(K) \to (F, \mathcal{T}_{\|\cdot\|}$.

By exercise 5.1. we have only to check that T is continuous in $\mathcal{T}_E \to \tau_F$, whichmeans that in the set $C^\infty(K)$ the norm itopology is finer i than pointwise topology. That is true!

So inkluusion mapping T is continuous $C^\infty(K) \to (F, \mathcal{T}_{\|\cdot\|}$ so the norm in F is continuous in th etoplogy of $C^\infty(K)$. So there exists l $\lambda > 0$ and a (semi)norm $p_{n,K}$,such that

$$\|\cdot\|_F \leq \lambda \|\cdot\|_n.$$

So in $E = C^\infty(K)$

$$\mathcal{T}_F \subset \mathcal{T}_{p_{n,K}}$$

. All in all in E

the subspace topology by $F \subset$ the subspace topology by $\mathcal{C}^k_K \subset \mathcal{C}^\infty_K$:s original topology. This implies $\mathcal{C}^k(K)$ completion (same as closure) of $(\mathcal{C}^\infty(K)$: in $\mathcal{C}^k(K) \subset$ completion/closure of $\mathcal{C}^\infty(K)$ in F norm topology . $\square$