Exercise help set 4 **Topological Vector Spaces**

4.2. *Prove Mazur's theorem assuming Hahn's and -Banach's theorem to be true. Don't use the axiom of choice again.*

Assume that $\emptyset \neq A \subset E$ open and convex and $M = x + F \subset E$, missä F a subspace. Assume that $A \cap M = \emptyset$.

Without loss of generality $0 \in A$. Assume that $p : E \to \mathbf{R}$ is the gauge of A. It is not necessarioly a seminorm (A not always balanced), but it is subadditive positively homogeneous function like in the full Hahn and Banach theorem. In the subspace $\langle M \rangle = \{x\} \oplus F$ there is a linear form defined by $f(\lambda x \oplus y) = \lambda$ and $f(x) = 1$ in M and therefore $|f(x)| \leq p(x)$ everywhere in $\{x\} \oplus F$ because $\{x\} \oplus F$ does not intersect A. By the Hahn and Banach theorem there exists a continuation of f toa lin form f in E , for which $|f| \leq p$. Because $p < 1$ in teh open set A, the set A does not intersectthe hyperplane $\{x \in E \mid f(x) = 1\}$. □

4.3. *Let E be a topological vector space and $S = \{x_\alpha \mid \alpha \in I\} \subset E$. We call the set S* topologically free, *or* topologically linearly independent, *if for all $\alpha \in I$ we have $x_\alpha \notin \overline{\langle S \setminus \{x_\alpha\} \rangle}$. Prove that if E is locally convex, then $S = \{x_\alpha \mid \alpha \in I\} \subset E$ is topologically free if and only of there exists a family of linear forms $S = \{f_\alpha \mid \alpha \in I\} \subset E^*$, that for all $\alpha \in I$*

(1) *f_α is continuous*
(2) $f_\alpha(x_\alpha) = 1$
(3) *$f_\alpha(x_\beta) = 0$ for all $\beta \in I \setminus \{\alpha\}$.*

Assume first that the forms f_α exist. If there exists $\alpha \in I$ such that $x_\alpha \in \overline{\langle S \setminus \{x_\alpha\} \rangle}$ then $f_\alpha(x_\beta) = 0$ for all $\beta \neq \alpha$, so by linearity $f_\alpha(x) = 0$ for all $x \in \langle S \setminus \{x_\alpha\} \rangle = \{$finite linear combinations of the vectors x_β $(\beta \neq \alpha)\}$. In particular $0 = f(x_\alpha) = 1$. TYhis contradiction proves the first statement.

Assume next that S is topologically free. This means that for all $\alpha \in I$ there exists

$$x_\alpha \notin \overline{\langle S \setminus \{x_\alpha\} \rangle}.$$

The subspace $\overline{\langle S \setminus \{x_\alpha\} \rangle}$ is closed, so by a well known (easy) corollary of Hahn and Banach there exists a continuous linear form $f_\alpha : E \to \mathbf{K}$, for which $f(x_\alpha) = 1$ and $f_\alpha = 0$ in teh subspace $\overline{\langle S \setminus \{x_\alpha\} \rangle}$. This is what we wanted.

4.4. *Let E be a complex topological vector space, $H = \{x \in E \mid f(x) = 0\}$ a hyperplane, so f is $\mathbf{C}$–linear $E \to \mathbf{C}$. Let $f_\mathbf{R}$ be the real part of f , which is defined as $f_\mathbf{R}(x) = \operatorname{Re} f(x)$. Prove that the subset $H_\mathbf{R} = \{x \in E \mid f_\mathbf{R}(x) = 0\}$ is a hyperplane in the real topological vector space E, which we may denote by $E_\mathbf{R}$. Prove also that $H = H_\mathbf{R} \cap (iH_\mathbf{R})$.*

a) It is sufficient to prove that $f_\mathbf{R}(x) = \operatorname{Re} f(x)$ is real linear $E \to \mathbf{R}$:

$$f_\mathbf{R}(x+y) = \operatorname{Re} f(x+y) = \operatorname{Re}(f(x) + f(y)) = \operatorname{Re}(f(x)) + \operatorname{Re} f(y)) = f_\mathbf{R}(x) + f_\mathbf{R}(y)$$

$$f_\mathbf{R}(\lambda x) = \operatorname{Re} f(\lambda x) = \operatorname{Re}(\lambda f(x)) \overset{*}{=} \lambda \operatorname{Re} f(x) = \lambda f_\mathbf{R}(x),$$

(at * notice: λ is real (!)).

b) $H = H_{\mathbf{R}} \cap (iH_{\mathbf{R}})$, sillä

$$\begin{aligned} x \in H &\implies f(x) = 0 \iff \operatorname{Re} f(x) = 0 \text{ and } \operatorname{Im} f(x) = 0 \\ &\iff x \in H_{\mathbf{R}} \text{ and } \mathrm{f}(-ix) = 0 \\ &\iff x \in H_{\mathbf{R}} \text{ and } (-ix) \in H_{\mathbf{R}} \\ &\iff x \in H_{\mathbf{R}} \text{ and } x \in iH_{\mathbf{R}}. \end{aligned}$$

4.5. *Prove that if E and F are Fréchet spaces and $T : E \to F$ is linear, then the graph $\operatorname{Gr} T$ is closed if and only if*

$$(x_n, f(x_n)) \to (0, y) \implies y = 0.$$

(How about more generall set-ups?)

In th eproduct topology $(x_n, T(x_n)) \to (0, y) \iff x_n \to 0$ and $T(x_n) \to y$.

a) If the graph is closed, then, by the closed graph theorem, T on continuous, so $x_n \to 0 \implies T(x_n) \to T(0) = 0$. If $x_n \to 0$ and $T(x_n) \to y$, then$0 = y$, since as a Fréchet space F is Hausdorff, so limits are unique.

b) To get back, assume that $(x_n, T(x_n)) \to (0, y) \implies y = 0$. Assume that $(x, y) \in \overline{\operatorname{Gr} T}$. Since E and F are metric, also the product space is metric so there exists in $\operatorname{Gr} T$ a sequence $(x_n, y_n) \to (x, y)$ same as $(x_n, T(x_n)) \to (x, y)$. Let us apply the assumption to $(x_n - x)_{\mathbf{N}}$, for which we know at least that $(x_n - x) \to 0$. Since $T(x_n - x) = T(x_n) - Tx = y_n - Tx \to y - Tx$, the assumptiongives $y - Tx = 0$. So $T(x) = y$, and $(x, y) \in \operatorname{Gr} T$, which is now proven closed.

4.6. *The direct linear algebraic sum $E = M \oplus N$ of two subspaces of a topological vektor space is called a* topological direct sum*, if its subspace topology is the same as its product topology, i.e. the mapping $(x, y) \mapsto x + y$ is a homeomorphism between the product space $M \times N$ and the subspace $M \oplus N \subset E$. Sometimes this is expressed by calling N a* topological supplement *of M. Let π be the projection from $E = M \oplus N$ to its subspace M in the diredction N, ie. $\pi(x + y) = x$, for $x \in M$ and $y \in N$.*

a) Prove that if M and N are topological, and linear subspaces of E, then $E = M \oplus N$ is a topological direct sum if and only if π is continuous.

b) Prove that if E is a Fréchet space, and if both M and N are closed subspaces, then π is continuous and $M \oplus N$ a topological direct sum. (Is it sufficient to assume that one of the two subspaces is closed?)

a) Assume that the direct sum is topological, ie as topological spaces $E = M \oplus N = M \times N$. Then π is continuous by the definition of the product topology.

To get back, assume that π continuous. In this case also the projection to N , same as $Id_{M \oplus N} - \pi$ is continuous. The space $M \oplus N$ has a topology in which the projections are continuous, so the topology is finer than the product topology. On the other hand, the sum mapping $E \times E \to E : (a, b) \mapsto a + b$ is continuous, so also its restriction $M \oplus \{0\} \times \{0\} \oplus N \to M \oplus N$ which is the linear isomorphism $M \times N \to M \oplus N$ is continuous, so the topology of the direct sum is coarser than the product topology. They coincide!

b) If E is a Fréchet space, and if both M and N are closed subspaces then both M and N are Fréchet spaces, so also their product space $M \times N$ and, by assumption also $E = M \oplus N$. We know already that the linear isomorphism $M \times N \to M \oplus N$ is a continuous surjection, so by the open mapping theorem it is a homeomorphism.

4.7. *(continuation) c) Let $T : E \to F$ be a continuous linear mapping, where E and F are topological vector spaces. Prove that T has a continuous linear* right inverse*, (same as a continuous, linear $S : F \to E$, for which $F \circ G = id_F$, if T is an open surjection) and the kernel* $\ker T \subset E$ *has a topological supplement M.*

By assumption $E = M \oplus N$ where $N = \ker T$. Define $S(T(m \oplus n)) = m$. This is well defined, since $T(m \oplus n) = T(m \oplus n)$ for all $n, n' \in N = KerT$. Assume that the kernel $N = \ker T \subset E$ has a topological supplement M. So $E = M \oplus N \sim M \times N$. Now linear mapping $J = \phi \big|_M : M \to E/N$ is a linear isomorphism, since it is injective (If $J(x) = 0_{E/N}$, then $x \in M \cap \ker \phi = M \cap N = \{0\}$.) and surjective (For every $v + N \in E/N$ same as $v = (x, y) \in E = N \oplus N$ on $v + N = \phi(v + n)$ for all $n \in N$. In particular $v + N = \phi(v - y) = \phi(x)$, where $x \in M$.

So the mapping $T \big|_M$ has a linear inverse $S : F \to M \subset E$. But continuous? Assume that $A \subset E = M \oplus N$ open. Then $A \cap M$ is open in a M and $(A \cap M) \times N$ is open in the product space $M \times N$ so $(A \cap M) \oplus N$ is open in $E = M \oplus N$. Since T is open by assumption, the set $T((A \cap M) \oplus N) \subset F$ is open. But since $N = \ker T$, we have $T((A \cap M) \oplus N) = T(A \cap M) = S^{-1}(A \cap M) = S^{-1}(A)$, which was to be proven open.

a) Let E be a vector space, and $\mathcal{B} = \{A \subset E \mid A \text{ absobing, balanced and convex}\}$. Prove that $\mathcal{B}$ defines a locally convex topology $\mathcal{T}$, in fact the finest possible locally convex topology on E.

b) Prove that if F is a locally convex space, then every linear mapping $E \to F$ is continuous.

a) Every family of absorb., bal. and conv defines some semonorms and a locally convexn top $\mathcal{T}$ in E. Assume that $\mathcal{T}'$ is a lokaali convex topologia in E. It has a neighbourhood basis of the origin consisting of barrels. These are – by assumption – neighbourhoods of the origin in the topology $\mathcal{T}$, so $\mathcal{T}$ is finer than $\mathcal{T}'$.

b) Assume that $T : E \to F$ is a linear mapping and $p : F \to \mathbf{R}$ a continuous seminori. Then $p \circ T : E \to \mathbf{R}$ oia a seminorm in the space E, so its (0-)balls are abs., bal. and convex, hence neighbourhoods of 0. So $p \circ T$ is continuous, and that proves continuity of the lin mapping T. (In fact $p \circ T$ is continuous,independent of whether p is continuous or not.)

From last week:

4.8. *An example of a subset of a locally convex space which is sequentially complete but not complete: $E = \mathcal{F}([0,1], \mathbf{R}) = \mathbf{R}^{[0,1]} = \{$allo functions $[0,1] \to \mathbf{R}\}$. Topology of pointwise convergence ie seminorms $p_x = |f(x)|$. $M = \{f \in E \mid f(x) \neq 0$ for at most countably many $x \in [0,1]\}$.*

M is obviously a E (top n and lin) subspace.

a) Consider a Cauchy sequence $(f_n)_{\mathbf{N}}$ in M. In the topology of pointwise convergence, Cauchy means that for all $t \in [0,1]$ and $\epsilon > 0$ there exists $n_0 = n_{\epsilon,t} \in \mathbf{N}$ such that $|f_n(t) - f_m(t)| < \epsilon$, kun $n, m \geq n_0$. For each t the sequence of real numbers $(f_n(t))_{\mathbf{N}}$ is Cauchy , soit converges: $f_n(t) \to f(t) \in \mathbf{R}$. This defines a function $f \in E$. Evidently $f_n \to f$ pointwise, Prove $f \in M$: Let us write

$$H_n = \{x \in \mathbf{R} \mid f_n(x) \neq 0\}.$$

Every H_n is countable, so also

$$H = \bigcup_{n\in\mathbf{N}} H_n = \{x \in \mathbf{R} \bigm| \exists n : f_n(x) \neq 0\}$$

is countable. Of course $f(t) = 0$ for all t, for which every $f_n(t) = 0$, so $f \in M$.

b) Next find a non-convergent Cauchy-filter $\mathcal{F}$ in M.

Idea: M is not closed. The constant function $g(t) = 1$ is in the closure, so theree is a filter basis consisting of subsets of $M$and converging to F in E. This may be what we want?.in tehe space E, it is a Cauchy-filter, jso its *trace*i $\mathcal{F} = \{A \cap M \bigm| A \in \mathcal{F}\}$ in M is a Cauchy-filter in Mi or contains the empty set. Let us disprove the second alternative. In the pointwiise topology $U \in \mathcal{U}_{g,E} \iff U \supset \{f : [0,1] \to \mathbf{R} \bigm| |f(t_i) - 1| \leq \epsilon\}$ for some finitely many $t_1, \dots t_k \in [0,1]$ and an $\epsilon > 0$. This set contains a function in M, for example the function with value 1 at $t_1, \dots t_k$ and 0 elsewhere. If $\mathcal{F}$ would converge to some $h \in M$, then it would be a filter basisin E converging both to h and to f which is impossible since limits are unique in Hausdorff spaces and E is T_2.

4.9. If You like to do more. *Let $K \subset \mathbf{R}^n$ be compact. In the space*

$$E = \mathcal{C}_c^\infty(K) = \{f : \mathbf{R}^n \to \mathbf{R} \bigm| f \in \mathcal{C}^\infty,\ \operatorname{supp} f \subset K\}$$

use the semonorms

$$q_\alpha(f) = \sup_{x\in K} \left| \left(\frac{\partial}{\partial x}\right)^\alpha f(x) \right|,$$

where $\left(\frac{\partial}{\partial x}\right)^\alpha f(x)$ is the (higher) partial derivative corresponding to the multi-index $\alpha \in \mathbf{N}^n$ (You can take $\mathbf{R}^1$ and usual higher derivatives - it makes no difference) . Write $\mathcal{Q} = \{q_\alpha \bigm| \alpha \in \mathbf{N}^n\}$. Prove that a $(E, \mathcal{Q})$ is Fréchet space. (loc-con, metr, compl)

For simplicity, consider the one dimensional case, where $\left(\frac{\partial}{\partial x}\right)^\alpha f$ is simply the α:th derivative $f^{(\alpha)}$.

Of course, the seminorms $p_{\alpha,K}(f) = \sup_K |f^{(\alpha)}|$ define a lc topology, evidently metrizable. Completeness? Take a Cauchy sequence $(f_i)_{i\in\mathbf{N}}$ in E. For each $\epsilon > 0$ and $n \in \mathbf{N}$ there exists an $N_{n,\epsilon} \in \mathbf{N}$ such that $p_n(f_i - f_j) \leq \epsilon$ for $i, j \geq N_{n,\epsilon}$. In particular, the sequence f_n is Cauchy in the sup-norm p_0, so it converges uniformly in K to some f. Similarly, the derivatives f' converge uniformly in K to some function g. By basic analysis, $g = f'$. Similarly for higher derivatives. Clearly $f_n \to f$ in E.