JYVÄSKYLÄN YLIOPISTO

MATEMATIIKAN JA
TILASTOTIETEEN LAITOS

Exercise help set 3 **Topological Vector Spaces**

3.1. *Every compact set $K \subset \mathbf{R}^n$ defines a seminorm $p_K(f) = \sup f(K)\ (= \max f(K))$ in the space $E = \mathcal{C}(\mathbf{R}^n) = \{f : \mathbf{R}^n \to \mathbf{R} \mid f$ is continuous$\}$. These seminorms give rise to a locally convex topology $\mathcal{T}$.*

a) Is $\mathcal{T}$ a Hausdorff-topology?

b) Is the sequence $f_n(x) = \frac{1}{n}e^x$ convergent in the topology $\mathcal{T}$?

c) Does there exist in E a normi, giving the topology $\mathcal{T}$, eli onko E normeerautuva? (Is $\mathcal{T}$ normable?) (answer : no. Why?)

a) Yes T_2. Check at the origin! Take any $f \in E \setminus \{0\}$ This means $f \neq 0$. By continuity there exists $\epsilon > 0$ and a compact interval $K \subset \mathbf{R}$, so that $f(x) > 3\epsilon$. Now $B_{p_K,0,\epsilon} \cap B_{p_K,f,\epsilon} = \emptyset$.

b) Converges to 0. The semiballs $B = B_{p_K,0,\epsilon}$ form a neighbourhood basis of the origin. Let $B = B_{p_K,0,\epsilon}$. Now $p_K(f_n) = \sup f(K) = \frac{\sup_K(e^x)}{n} \to 0$, so there exists $n_0 \in \mathbf{N}$ such that $f_n \in B$, whenever $n \geq n_0$.

c) No. If it were true, then some norm $\|\cdot\|$ give the same topology $\mathcal{T}$.

Then in particular the norm $\|\cdot\|$ is continuous mapping, so the unit ball conatins a neighbourhood in the topology $\mathcal{T}$. and that containsa semiball $B = B_{p_K,0,\epsilon}$,since the semiballs $B = B_{p_K,0,\epsilon}$ form a neighbourghood basis of the origin. Then $p_K(f) < \epsilon \implies \|f\| \leq 1$ and so $p_K(f) < \frac{\epsilon}{n} \implies \|f\| \leq \frac{1}{n}$, so $p_K(f_n) \to 0 \implies \|f_n\| \to 0 \implies f_n \to 0$,which is untrue.

3.2. *The seminorms $p_n(f) = \sup_{0 \leq t \leq 1} |f^{(n)}(t)|$ $(n = 0, 1, 2, \dots)$ define a cocally convex topology $\mathcal{T}$ in the space $E = \mathcal{C}^\infty([0,1]) = \{f : [0,1] \to \mathbf{R} \mid f$ is infintely many times derivable$\}$. For $f \in E$, denote*

$$Tf(x) = \int_0^x f(t)\,dt.$$

T so is a linear mapping (also called an operaator or eli transformation) $E \to E$.

a) Is T continuous?

b) Is the topology $\mathcal{T}$ normable?

a) T is continuous. The derivativews of the iomage function $g = Tf$ are evidently $g' = f$, $g'' = f', \dots, g^{(n)} = f^{(n-1)}$. So for all $n > 1$ we have $p_n(Tf) = p_n g = p_{n-1} f$, so we only have to check p_0.

$$p_0(Tf) = \sup |\int_0^x f(t)\,dt| \leq \sup |f| = p_0(f). \text{ OK!}$$

b) NO. If some norm $\|\cdot\|$ would give the same topology $\mathcal{T}$, then in particular this norm $\|\cdot\|$ is a continuous mapping, so $B_{\|.\|}$ contains a neighbourhood of the origin in $\mathcal{T}$, and that contains an intersection of finite many semiballs, so $B_{\|.\|} \supset \lambda \bigcap_{i=1}^m B_{p_{n_i}}$, $(n_1 < n_2 < \cdots < n_m\ (\in \mathbf{N}))$. So for some $\mu > 0$ we have

$$\|\cdot\| \leq \mu \max_{1 \leq i \leq n_m} p_{n_i}.$$

But by the antithesis that $\|\cdot\|$ alone defines the topology, $\{\|\cdot\|\}$ is a basis of continuous seminorms and so there exists a number $\lambda > 0$ such that $p_{n_m+1} < \lambda\|\cdot\|$. Combining the results gives

$$p_{n_m+1} < \lambda\|\cdot\| \leq \lambda\mu \max_{1\leq i\leq n_m} p_{n_i}.$$

This is impossible, since there exists a sequence $(f_\alpha)_{\alpha\in\mathbf{N}}$, for which $p_{n_i}(f_\alpha) \to 0$ for all $i = 1,\ldots,m$, whenever $\alpha \to \infty$, but not $p_{n_m+1}(f_\alpha) \to 0$. You can choose $f_\alpha(x) = \alpha^{-n_m-1}\sin(\alpha x)$, since $f_\alpha^{(n)}(x) = \alpha^{-n_m-1-n}\cdot r(x)$, where $r(x)$ is a biunded function, so $p_n(f_\alpha) = \sup_{[0,1]}|f_\alpha^{(n)}(x)| \to 0$, whenever $n \leq n_m$ and $\alpha \to \infty$, but does not converge to 0 whenever $n > n_m$. In particular $p_{n_i}(f_\alpha) \to 0$ for all $i = 1,\ldots m$ but not $p_{n_m+1}(f_\alpha) \to 0$.

3.3. Let E be a real locally convex space and $A \subset E$ concex. Prove that A is closed if and only if A is the intersection of some closed half spaces in E.

Since an intersection of closed sets is closed, it is sufficient to check the other side. Let A be convex and closed. The complement $C = \backslash A$ is open. Let $x \in C$. There exists an open neighbourhood U of the point x, such that $U \subset C$. Since E is locally convex U canbe chosen to be convex. Since E is real, Banach's separation theorem gives a continuous linear form f and a number $\alpha > 0$ (In fact you can take $\alpha = 1$) such that $A \subset H = \{x \in E \mid f(x) \leq \alpha\}$ and $U \cap H = \emptyset$, in particular $x \notin H$. THIS PROVES IT.

3.4. *Let E be a normed space. Prove that the norm $x \mapsto \|x\|$ is discontinuous E in the weak topology of E (weakly continuous). (The weak topology is defined by the seminorms $x \mapsto |\langle x, x*\rangle|$, where $x* \in E^*$ ={continuous lin forms}.)*

Is it lower semicontinuous? For this it is sufficient that it is the pointwise supremum of a family of continuous mappings.

a) If the norm would be continuous, then by the characterization of continuous seminorms, there would exist a number $n \in \mathbf{N}$ and continuous seminorms $|\langle\cdot, x_i^*\rangle|$, (i=1,...n), such that

$$\|\cdot\| \leq \sum_{i=1}^{n} |\langle\cdot, x_i^*\rangle|.$$

Now

$$x \in \cap_{i=1}^n \ker x_i^* \implies \|x\| = 0 \implies x = 0,$$

so the linear mapping $T : E \to \mathbf{K}^n : x \mapsto (\langle x, x_1^*\rangle, \ldots, \langle x, x_n^*\rangle)$ is injective, and therefore $\dim E \leq \dim \mathbf{R}^n = n < \infty$.

b) By a well known corollary to the Hahn and Banach theiorem

$$\|x\| = \sup_{x\in E^*\backslash\{0\}} \frac{|\langle x, x^*\rangle}{\|x^*\|},$$

which is a supremum of continuous mappings

$$x \mapsto \frac{|\langle x, x^*\rangle}{\|x^*\|}.$$

3.5. *Let (E,P) be a locally convex space. Prove that the sequence $(x_n)_{\mathbf{N}}$ in E is a Cauchy-sequence if and only if*

$$\forall p \in P \text{ and } \forall \epsilon > 0\ \exists n_0 \in \mathbf{N} \text{ s.e. } q, r \geq n_0 \implies p(x_q - x_r) \leq \epsilon.$$

By definition, the sequence (x_n) is Cauchy in the topological space $E, \mathcal{T}$ if and only if the corr, which in turn means that for all $U \in \mathcal{U}$ there exists a number $n_0 \in \mathbf{N}$ such that $x_n - x_m \in U$ for all $n, m \geq n_0$.

Here (E,P) is locally convex avaruus, joten we may asssume that U is a basis set and of the form $U = \lambda \bigcap_{p \in \mathcal{H}} U_p$, where $\mathcal{H} \subset \mathcal{P}$ is finite set of basis seminorms. Now $x_n - x_m \in U$ in the definitionabove looks like saa $x_n - x_m \in \lambda \bigcap_{p \in \mathcal{H}} U_p$, same as

$$p(x_n - x_m) \leq \lambda\, \forall p \in \mathcal{H}.$$

1) (only if) If the sequence is Cauchy and $p \in P$ and $\epsilon > 0$, we choose $\mathcal{H} = \{p\}$ and $\lambda = \epsilon$, ao the definition give the claim.

2) (if) If the sequence fulfills the claim, and U is of the form $U = \lambda \bigcap_{p \in \mathcal{H}} U_p$, where $\mathcal{H} \subset \mathcal{P}$ is finite, then we choose for each $p \in \mathcal{H}$ a number n_p such that $p(x_n - x_m) \leq \lambda\, \forall n, m \geq n_p$. For $n, m \geq n_0 = \max\{n_p \mid p \in H\}$, we have $p(x_n - x_m) \leq \lambda\, \forall n, m \geq n_p$ same as $x \in \lambda \bigcap_{p \in \mathcal{H}} U_p$.

3.6. *Let $E = \prod_{i \in I} E_i$ be the product of topological vector spaces (product topology!) and $\pi_i : E \to E_i$ a standard projection ($i \in I$). Prove that $\mathcal{F}$ is a Cauchy filter in E if and only if every $\pi(\mathcal{F}) \subset E_i$ is a Cauchy filter in E_i. (Take $I = \{1, 2\}$ if you want an easy case)*

a) The product topology is the finest topology, where each projection $\pi_i : E = \prod_{i \in I} E_i \to E_i : x \mapsto x_i$ is continuous. Evidently they are linear surjections, so let us prove a slightly more general theorem: **in a tvs E the image of a Cauchy-filter $\mathcal{F}$ in a continuous linear mapping $T : E \to F$ is a Cauchy-filter.** Remeber, in any mapping the images of the sets belonging to a filter, forma a filter basis spanning what is sometimes called the image of the original filter. A surjection will give the whole filter. Let us verify the Cauchy condition Let $U \in \mathcal{U}_F$. Because $T^{-1}U \in \mathcal{U}_E$, there exists $M \in \mathcal{F}$, such that $M - M \subset T^{-1}U$, so $T(M)$ belongs too the image filter and $T(M) - T(M) = T(M - M) \subset T(T^{-1}U) \subset U$.

Next assume $\pi_i(\mathcal{F})$ is Cauchy. Let $U \in \mathcal{U}_E$ in the product topology. We caan assume that U is a basis set of the form

$$U = \prod_{j \in J} U_j \times \prod_{i \in I \setminus J} E_i,$$

where J is finite and $U_j \in \mathcal{U}_{E_i}$. VChoose for each $j \in J$ a set $M_j \in \pi_j(\mathcal{F})$, such that $M_j - M_j \subset U_j$. By the definition of theimage filter $M_j \supset \pi_j(N_j)$ for some $N_j \in \mathcal{F}$. (By surjectivity, one could take $M_j = \pi_j(N_j)$.) Take

$$N = \prod_{j \in J} N_j \times \prod_{i \in I \setminus J} E_i.$$

Evidently $N + N \subset U$, so we only have to verify $N \in \mathcal{F}$. Notice $N = \bigcap_{j \in J} \pi^{-1}(M_j) = \bigcap_{j \in J} \pi^{-1}(\pi(N_j)) = N \supset \bigcap_{j \in J} N_j$ and remebeer $N_j \in \mathcal{F}$ and that any filter contains the finite intersections and supsets of its elements. OK!

3.7. *(continue) Prove that $E = \prod_{i\in I} E_i$ is complete if and only if each E_i is complete.*

Completeness of E means that every Cauchy filter $\mathcal{F}$ converges ie. there exists $x \in E$, such that $\mathcal{F} \supset \mathcal{U}_x\,(= x + \mathcal{U}_0)$.

Let first every E_i be complete and $\mathcal{F}$ a Cauchy filter in teh product space E. Byy the previous exercise the image filters $\mathcal{F}_i = \pi_i(\mathcal{F})$ are Cauchy, so they converge: $F_i \supset \mathcal{U}_{x_i}$ for some $x_i \in E_i$. Let us verify $\mathcal{F} \to x$ same as $\mathcal{F} \supset \mathcal{U}_x = x + \mathcal{U}_{0,E}$, where $x = (x_i)_{i\in I} \in E$:

Let $U \in \mathcal{U}_x \subset E = E = \prod_{i\in I} E_i$. We can assume that U is a basis set of the form

$$U = x + \prod_{j\in J} U_j \times \prod_{i\in I\setminus J} E_i = x + \bigcap_{j\in J} \pi^{-1} U_j,$$

where J is finite and $U_j \in \mathcal{U}_{0,E_i}$. Kullakin $j \in J$ is $U_j \in \mathcal{F}_j = \pi_j(\mathcal{F})$, joten $\pi^{-1}U_j \in \mathcal{F}$ and siis $\bigcap_{j\in J} \pi^{-1}U_j \in \mathcal{F}$, joten $U \in x + \mathcal{F}$.

Let now $E = \prod_{i\in I} E_i$ be complete and $\mathcal{F}_j$ a Cauchy filter in E_j for some $j \in I$. Let for all $i \neq j$ $\mathcal{F}_i = \mathcal{U}_{E_i}$ and let $\mathcal{F}$ be a filter in E with basis sets $N_i = \pi^{-1}M_i$, $i \in I$, $M_i \in \mathcal{F}_i$. (Tcould be called a product of filters?.) By the previous ex, thi is Cauchy, since $\pi_i(\mathcal{F}) = F_i$ for all $i \in I$ and both $\mathcal{F}_j$ and each $\mathcal{U}_{E_i}$ are convergent, hence Cauchy. So by assumption, $\mathcal{F}$ converges: $\mathcal{F} \supset x + \mathcal{U}_E$ for soe $x = (x-i)_{i\in I} \in E$. Mow verify $\mathcal{F}_j \to x_j$ same as $\mathcal{F}_j \supset x_j + \mathcal{U}_{E_j}$. Let $x_j + U_j \in x_j + \mathcal{U}_{e_j}$. Now $x + \pi_j^{-1}(U) \in x + \mathcal{U}_E \subset \mathcal{F}$. Siis

$$x_j + U_{E_j} \supset \pi_j(x + \pi_j^{-1}(U)) \in \pi_j(\mathcal{F}) = \mathcal{F}_j,$$

so $x_j + U_{E_j} \in \mathcal{F}_j$, and therefore $x_j + \mathcal{U}_{E_j} \subset \mathcal{F}_j$.

3.8. *Let*

$$E = \{f \in \mathcal{C}[0,1] \bigm| \exists \epsilon_f > 0 \text{ such that } f(t) = 0\ \forall\, 0 \le t \le \epsilon(f)\}$$

withthe norm $\|f\| = \sup|f|$. Is

$$T = f \in E \bigm| |f(\tfrac{1}{n}) \le \tfrac{1}{n}\ \forall n \in \mathbf{N}^*\}$$

a barrel? Is it a neighbourhood of the origin? (Why do I ask?)

a) The set is absorboing, since if $f \in E$, we choose $n_\epsilon = \max\{n \in \mathbf{N} \bigm| \frac{1}{n} \ge \epsilon\}$, and find $\lambda f \in T$ at least whenever

$$|\lambda| \le \frac{1}{n_\epsilon}\Big(\max_{1\le n\le n_\epsilon} |f(\tfrac{1}{n})|\Big)^{-1}.$$

Thew set T is obviously balanced and convex. It is also closed (so a barrel), since its complement is open; Let $f \in E \setminus T$. Then there exists $n \in \mathbf{N}$, for which $|f(n)| > \frac{1}{n}$. Take $r = (|f(n)| - \frac{1}{n})$. Then the open ball $B(f,r) = \{g \in E \bigm| \|f-g\| < r$ is included in the complement E_T.

The set T is not a neighbourhood of the origin. If it were, it would contain a ball $B(0,\frac{1}{n})$, but it does not! Look at any continuous, monotonously increasing function which is constant 0 on $[0,\frac{1}{2n}]$ and the constant $\frac{1}{n}$ on $[\frac{1}{2n},1]$.

So we have proven that the normaed space E is not barreled. By the barrel theorem (from Baire) we caninterfere that it is not complete.

3.9. *An example of a subset of a locally convex space which is sequentially complete but not complete:* $E = \mathcal{F}([0,1], \mathbf{R}) = \mathbf{R}^{[0,1]} = \{$*allo functions* $[0,1] \to \mathbf{R}\}$. *Topology of pointwise convergence ie seminorms* $p_x = |f(x)|$. $M = \{f \in E \bigm| f(x) \neq 0$ *for at most countably many* $x \in [0,1]\}$.

Solution due next week

3.10. If You like to do more. *Let* $K \subset \mathbf{R}^n$ *be compact. In the space*

$$E = \mathcal{C}_c^\infty(K) = \{f : \mathbf{R}^n \to \mathbf{R} \bigm| f \in \mathcal{C}^\infty,\ \operatorname{supp} f \subset K\}$$

use the semonorms

$$q_\alpha(f) = \sup_{x \in K} \left| \left(\frac{\partial}{\partial x} \right)^\alpha f(x) \right|,$$

where $\left(\frac{\partial}{\partial x}\right)^\alpha f(x)$ *is the (higher) partial derivative corresponding to the multi-index* $\alpha \in \mathbf{N}^n$ *(You can take* $\mathbf{R}^1$ *and usual lhigher derivatives - it makes no difference) . Write* $\mathcal{Q} = \{q_\alpha \bigm| \alpha \in \mathbf{N}^n\}$. *Prove that a* $(E, \mathcal{Q})$ *is Fréchet space. (loc-con, metr, compl)*

Solution due next week, if ever.