JYVÄSKYLÄN YLIOPISTO

MATEMATIIKAN JA
TILASTOTIETEEN LAITOS

Harjoitukset 5 **Topologiset vektoriavaruudet**
tiistai 26.10.2010 14.30-16.00 MaD-355

Bairen- Fréchet.

5.1. Assume $(E, \mathcal{T}_E)$ and $(F, \mathcal{T}_F)$ are Fréchet spaces and in the space F there also is another Hausdorff–topology τ_F, coarser than $\mathcal{T}_F$. Assume $T : E \to F$ linear.

Prove that if T is continuous $\mathcal{T}_E \to \tau_F$, then it is continuous $\mathcal{T}_E \to \mathcal{T}_F$ (Looka ta the graph!).

5.2. Assume $(E, \mathcal{T}_E)$ and $(F, \mathcal{T}_F)$ Fréchet spaces and assume $(X, \mathcal{T}_X)$ is a Hausdorff-space . Assume $T : E \to F$ linear. Prove that T is continuous, if there exists a continuous injection $f : F \to G$, such that $f \circ T$ is continuous. (Idea: proj topology)

Function spaces . Solutions to some problems in Exx 3 and 4.

5.3. Assume $k \in \mathbf{N} \cup \{\infty\}$. In the space $E = \mathcal{C}^k = \mathcal{C}^k(\mathbf{R}) = \{f : \mathbf{R} \to \mathbf{R} \bigm| f$ is k times differentiable$\}$ the standard topology, also called the topology of *compact* $\mathcal{C}^k$*–convergence* is the lokaalikonveksi topology, given by the seminorms

$$\left| \left(\frac{\partial}{\partial x} \right)^{\alpha} f(x) \right| = \sup_{x \in K} |f^{(\alpha)}(x)|.$$

Prove that every $\mathcal{C}^k$ is metrisable and Hausdorff.

5.4. Prove that every $\mathcal{C}^k$ $(k \in \mathbf{N})$ is (seq)complete , so it is Fréchet. (Banach? Is there a continuousa norm ?)

5.5. Prove that $\mathcal{C}^\infty$ is (seq)complete , so it is Fréchet. (Banach? Is there a continuousa norm ?)

5.6. Assume $K \subset \mathbf{R}$ compact and $k \in \mathbf{N} \cup \infty$. (One K fixed.) Prove that alis paces $\mathcal{C}^k(K) = \{f \in \mathcal{C}^k \bigm| \operatorname{supp} f \subset K\}$ are (jono)complete, so Fréchet. (Banach? Is there a continuousa norm ?)

5.7. Prove that the completion (same as closure!) of $\mathcal{C}^k(K)$ in $\mathcal{C}^k(K)$ is is $\mathcal{C}^\infty(K)$. This means that $\mathcal{C}^\infty(K)$ is dense in $\mathcal{C}^k(K)$ (which is complete).

5.8. Prove that the standard topology of $\mathcal{C}^k$ is also defined by the seminorms $\mathcal{Q}$, of seminorms

$$q_K(f) = \int_K |f^{(n)}(x)|\, dx,$$

where $K \subset \mathbf{R}$ is compact and $n \in \mathbf{N}$. Vihje: $f(x) = \int_x^{x+1} \left((t - x - 1) f'(t) + f(t)\right)\, dt$.

Remark. In analysis, often n-dimensional versions of the spaces $\mathcal{C}^k$ are used: $E = \mathcal{C}^k = \mathcal{C}^k(\mathbf{R}^n) = \{f : \mathbf{R}^n \to \mathbf{R} \bigm| f \text{ is } k \text{ times diff}\}$ and topology byseminorms

$$p_\alpha(f) = \sup_{x\in K} \left| \left(\frac{\partial}{\partial x}\right)^\alpha f(x) \right|,$$

where $K \subset \mathbf{R}^n$ is compact and $\left(\frac{\partial}{\partial x}\right)^\alpha f(x) = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f(x)$ is the partial derivative correspinds to multi-indeksiä $\alpha = (\alpha_q, \dots, \alpha_n) \in \mathbf{N}^n$ (so $(\alpha_1 + \dots + \alpha_n = \alpha)$. More generally, the set $\mathbf{R}^n$ can be replaced by any open set $\Omega \subset \mathbf{R}^N$. All this brings no essential change to what was done above. .

Combination.

5.9. Assume $K \subset \mathbf{R}^n$ compact set and $F \subset \mathbf{R}^K$ Banach space , whose elementscalled vectors, or points) are funstions $K \to \mathbf{R}$ ie $E\pounds isavectorsubspaceof \mathbf{R}^K$. Assume , that the topology in F is finer than the topology of pointwise convergence, which is the product topology in / from $\mathbf{R}^K$y. Assume that $\mathcal{C}^\infty(K) \subset F$.

We prove , that there exists la number $k \in \mathbf{N}$, such that $C^k(K) \subset F$.

a) Apply ex ä 5.1. choosing: for E: the space C^∞ vwith its standard toplogy now called $\mathcal{T}_E$. For F we choose the norm topology $\mathcal{T}_F = \mathcal{T}_{\|\cdot\|}$ is inclusion $x \mapsto x$.

Prove that the inclusion mapping T is continuous $\mathcal{T}_E \to \mathcal{T}_F$ same as a mapping $C^\infty(K) \to (F, \mathcal{T}_{\|\cdot\|}$.

b) find out, that there exists a number $\lambda > 0$ and a (semi)norm $p_{n,K}$, such that $\|\cdot\|_F \le \lambda p_{n,K} = \lambda \|\cdot\|_n$. Check (or rememeber) , that the continuous seminormi $f \mapsto p_{n,K}(f) = \sup_{x\in K} |f^{(n)}(x)|$ is in fact a norm in $E = C^\infty(K)$. So in $E = C^\infty(K)$ we have $\mathcal{T}_F \subset \mathcal{T}_{p_{n,K}}$ and all in all

The subspace topology from F l $\subset$ topology from $\mathcal{C}^k_K \subset$ original topology in $\mathcal{C}^\infty_K$

So. explain why

The completion of $\mathcal{C}^k(K) = (\mathcal{C}^\infty(K)$: in $\mathcal{C}^k(K)$:ssa) $\subset$ the completion of $\mathcal{C}^\infty(K)$ in theoriginal norm of F. Are we done? $\square$