GROUPS AND THEIR REPRESENTATIONS - FIFTH PILE

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32. Representations of the group $SL_2(\mathbb{R})$

Example 32.1. Recall

$$SL_2(\mathbb{R}) = \left\{ A = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \middle| \det A = xw - yz = 1 \right\} \text{ and}$$

$$sl_2(\mathbb{R}) = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \operatorname{Tr} A = a + d = 0 \right\}$$

We have (in the exercises) already found the following irreducible representations for $SL_2(\mathbb{R})$:

(1) The trivial representation; one-dimensional,
(2) The tautological representation; one-dimensional,
(3) The symmetric powers of the above $Sym^d(\mathbb{R}^2)$, where $d = 2, 3, \ldots$.

we will prove that there ore no other irreducible representations. This is far from obvious. The idea of the proof is to use the fact that a representation of a Lie group is irreducible (if and) only if its derivative is irreducible as a lie algebra representation. We will prove that the lie algebra $sl_2(\mathbb{R})$ has no other irreducible representations except the derivatives of the group representations listed above. In fact we will classify the irreducible representations of the corresponding complex lie algebra $sl_2(\mathbb{C})$.

Proposition 32.2. *The derivative of an irreducible finite dimensional representation ρ of a Lie group G, at the neutral element, is an irreducible representation of the group's lie algebra $d_e\rho : \mathcal{G} \to gl(V)$.* [1]

CORRECT THE FINNISH TEXT!

[1] Irreducibility of ρ and irreducibility of its derivative are in fact equivalent, but we will need only this half of the statement here.

Todistus. Let $\rho : G \to GL(V)$ be an irreducible representation of the lie group G. If $d_e\rho : \mathcal{G} \to gl(V)$ were reducible, it would have a proper nonzero sub-representation i.e. a proper nonzero subspace W, invariant under actions of elements in $\mathcal{G}$.

One can show[2] that W is a subrepresentation of ρ also which is impossible. □

The next task b is to find the derivatives of the group representations listed above. For brevity, we denote the tautological representation of $SL_2(\mathbb{R})$ by $Sym^1(\mathbb{R}^2)$ and the trivial representation by $Sym^0(\mathbb{R}^2)$. The standard basi vectors of $\mathbb{R}^2$ are $e_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $e_2 = \begin{bmatrix}0\\1\end{bmatrix}$, so the basis vectors of the symmetric product $Sym^d(\mathbb{R}^2)$ are $e_1^d, e_1^{d-1}e_2, e_1^{d-2}e_2^2, \dots, e_1e_2^{d-1}$ and e_2^d, and the matrix $A \in SL_2(\mathbb{R})$ acts on the basis vector $e_1^{d-i}e_2^i$ by

$$\begin{bmatrix}x & y\\ z & w\end{bmatrix}(e_1^{d-i}e_2^i) = (Ae_1)^{d-i}(Ae_2)^i = (xe_1 + ze_2)^{d-i}(ye_1 + we_2)^i.$$

The derivative of this is found by repeatedly using the formula for the derivative of a bilinear mapping, and we arrive at the following lemma:

Lemma 32.3. *Let V and W be representations of the Lie group G.*

(a) Let $V \otimes W$ be their tensor product, i.e. $g(v \otimes w) = gv \otimes gw$ for all $g \in G, v \in V, w \in W$. Taking the derivative at the neutral element gives the corresponding lie algebra representation

$$X(v \otimes v) = Xv \otimes w + v \otimes Xv,$$

where each X is the derivative of the corresponding Lie group representation.

(b) Let $Sym^d(V)$ be a symmetric power of the representation V. Taking the derivative at the neutral element gives the corresponding lie algebra representation

$$X(e_1^{a_1} \cdot e_2^{a_2} \dots e_d^{a_d}) = \sum_{k=1}^{d} a_k\, e_1^{a_1} \cdot e_2^{a_2} \dots e_k^{a_k - 1} \dots e_d^{a_d} \cdot Xe_k,$$

where possible symbols e_j^0 stand for nothing.

[2] Find a proof!

Todistus. (a) is a consequence of the bilinear mapping derivation formula. Väite (b) follows from (a) and a simple factor space argument. □

Next we use the lemma to determine the derivative of the second symmetric power of the tautological representation of $GL_2(\mathbb{R})$. The lie algebra $sl_2(\mathbb{R})$ has the generators $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and as a vector space it has a basis $\{X, Y, H\}$, where $H = [X, Y] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The representation $Sym^2(\mathbb{R}^2)$ of $sl_2(\mathbb{R})$ is determined by the action of these basis vectors on the basis vectors of the representation space $Sym^2(\mathbb{R}^2)$. (In fact it would be sufficient to consider the action of the generators X and Y, but it turns out to be easier to take all three.) The basis vectors of $Sym^2(\mathbb{R}^2)$ are e_1^2, $e_1 \cdot e_2$ and e_2^2. Let us calculate the actions by the previous lemma **??**.

$$\begin{aligned} X(e_1^2) &= 2e_1 \cdot Xe_1 = 2e_1 \cdot 0 = 0 \\ X(e_1^2) &= Xe_1 \cdot e_2 + e_1 \cdot Xe_2 = 0 \cdot e_2 + e_1 \cdot e_1 = e_1^2 \\ X(e_2^2) &= 2e_2 \cdot Xe_2 = 2e_2 \cdot e_1 = 2e_1 \cdot e_2. \end{aligned}$$

Expressing the images of the basis vectors in the basis gives the matrices of the action of X;

$$Mat(Sym^2(X)) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

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$$Mat(Sym^2(Y)) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

The action of the third basis vector H of the lie algebra $sl : 2(\mathbb{R})$ is the classical bracket of these actions o X and Y, but instead of calculating it by matrix algebra we can find out its action directly, which turns out to be useful in the later generalisation to higher powers; we can in fact calculate $Sym^d(H)$ for higher powers right now: The effect of H on any basis vector of the symmetric power is

$$\begin{aligned} H(e_1^{d-i}e_2^i) &= (d-i)e_1^{d-i-1}e_2^i He_1 + ie_1^{d-i}e_2^{i-1}He_2 \\ &= (d-i)e_1^{d-i-1}e_2^i e_1 - ie_1^{d-i}e_2^{i-1}e_2 \\ &= (d-i)e_1^{d-i}e_2^i - ie_1^{d-i}e_2^i \\ &= (d-2i)\, e_1^{d-i}e_2^i. \end{aligned}$$

In particular, for $d = 2$ we have

$$Mat(Sym^2(H)) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix},$$

a diagonal matrix. From the calculation above it is obvious that $Mat(Sym^d(H))$ will be diagonal in the above basis for any d. Also, it is clear that its diagonal elements form a finite arithmetic sequence with difference 2, like this: $d_{11} = d, d-2, d-4, \ldots, d_{dd}$.

Lemma 32.4. *The basis element* $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ *of* $sl_2(\mathbb{R})$ *acts diagonally not only in the in the tautological representation and its symmetric powers but in all other irreducible*[3] *representations as well.*

Todistus. To be added later, ??? □

Remark 32.5. It is a fact that all so called semi-simple lie algebras $\mathcal{G}$ (like $sl_n(\mathbb{R})$) have the following property: If any element $X \in \mathcal{G}$ acts diagonally in the tautological representation, then it will act diagonally in all other representations as well.

To prove that we have found all irreducible representations of $SL_2(\mathbb{R})$, we now only have to prove that there are no other irreducible lie algebra representations of $sl_2(\mathbb{R})$ except the symmetric powers studied above.

By algebraic completeness of the field $\mathbb{C}$ it is easier to study complex than real lie algebras. therefore we find it useful to *complexify* $SL_2(\mathbb{R})$ by the following construction:

Remark 32.6. Let V be a representation of the lie algebra $sl_2(\mathbb{R})$. Then "the same vector space with complex coefficients", $V \otimes \mathbb{C}$ is a representation of the corresponding complex lie algebra $sl_2(\mathbb{R}) \otimes \mathbb{C}$. (Take the same matrices!).

If $W$is a sub-representation of a representation V of the lie algebra $sl_2(\mathbb{R})$[4], then $W \otimes \mathbb{C}$ is a complex sub-representation of the representation $V \otimes \mathbb{C}$ of the complex lie algebra $sl_2(\mathbb{R}) \otimes \mathbb{C}$. In particular, if $V \otimes \mathbb{C}$ is irreducible, then also the original representation V is irreducible.[5]

[3]?

[4]This works for any lie algebra, in my opinion.

[5]The converse is false: We have constructed an irreducible representation, whose complexification was reducible.

So for proving that we have found all irreducible representations of $sl_2(\mathbb{R})$, it is enough to prove that their complexifications are the only irreducible representations of the complex lie algebra $sl_2(\mathbb{R})\otimes\mathbb{C}$ [6]. This is the next theorem.

Theorem 32.7. *The only finite dimensional irreducible representations of the lie algebra $sl_2(\mathbb{C})$ are the symmetric powers $Sym^d(\mathbb{C}^2)$ of the tautological representation.*

Todistus. The matrices of the symmetric powers are the same for corresponding real and complex representations, i.e. the ones studied above.

Let us consider any finite dimensional (irr?) representation V of the lie algebra $sl_2(\mathbb{C})$. In the tautological representation, the lie algebra $sl_2(\mathbb{C})$ is generated, even spanned by the matrices $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Since we expect H to act diagonally, let us consider it first. By the [7]Lemma**??**, the action is diagonal , so V splits into a finite direct sum $V = \oplus_{\alpha\in\mathbb{C}} V_\alpha$, where H acts as multiplication by α in each subspace H_α kertomisena luvulla α. This is expressed by calling V_α the *eigenspace* of H with eigenvalue α.

Next find the action of X in each V_α. We will prove that $Xv \in V_{\alpha+2}$ for $v \in V_\alpha$. Verifying this relies on a clever idea: Let $v \in V_\alpha$, so $Hv = \alpha v$.

$$HXv = [H, X]v + XHv,$$

but in the group $GL_2(\mathbb{R})$ we have

$$[H, X] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 2X,$$

so $[H, X] = 2X$ also holds for the representation matrices, in particular

$$HXv = [H, X]v + XHv = 2Xv + X\alpha v = (2 + \alpha)Xv.$$

Therefore $Xv \in V_{\alpha+2}$.

Similarly $Y : V_\alpha \to V_{\alpha-2}$. These observations have significant consequences:

[6]which is the same as $sl_2(\mathbb{C})$

[7]UNPROVED!

Let $\alpha_0 \in \mathbb{C}$ be an eigenvalue of H, so $V_{\alpha_0} \neq \{0\}$. Then the direct sum

$$\oplus_{m\in\mathbb{Z}} V_{\alpha_0+2m} \subset V$$

is a sub-representation of V, since both generators of the lie algebra $sl_2(\mathbb{R})$, X and Y map it onto itself. But by assumption the representation is irreducible, and $V_{\alpha_0} \neq \{0\}$. therefore

$$\oplus_{m\in\mathbb{Z}} V_{\alpha_0+2m} = V.$$

since V was assumed finite dimensional, every eigenspace V_{α_0+2m} of H is finite dimensional and only finite many are non zero. This means

$$V = V_\lambda \oplus V_{\lambda+2} \oplus \cdots \oplus V_{\lambda+2n}.$$

since the representation matrices are invertible, all $V_\lambda, V_{\lambda+2}, \ldots, V_{\lambda+2n} = V_\mu$ are n.[8]

Let $v \in V_\mu$. Then $\langle v, Yv, Y^v, \ldots, Y^n v\rangle = V$, since also this is an invariant subspace of the invariant representation, since both Y and H map it onto itself, and also X does the same, since

$$X(Y^p v) = p(\mu - p + 1)(Y^{p-1}v),$$

which can be proved by induction:

Initial case p=0: Observe that $XY^p v = XY^0 v = Xv = 0$, since we assumed $v \in V_\mu$.So the laim is true for $p = 0$.

Induction step: Assume

$$XY^{p-1}v = (p-1)(\mu - (p-2))Y^{p-2}v.$$

Use $Y : V_\alpha \longrightarrow V_{\alpha-2}$ to calculate:

$$\begin{aligned}
XY^p v &= XYY^{p-1}v \\
&= [X,Y]Y^{p-1}v + (XY - [X,Y])Y^{p-1}v \\
&= [X,Y]Y^{p-1}v + (YX)Y^{p-1}v \\
&= [X,Y]Y^{p-1}v + YXY^{p-1}v \\
&= HY^{p-1}v + Y(p-1)(\mu - (p-2))Y^{p-2}v \\
&= (\mu - 2(p-1))Y^{p-1}v + (p-1)(\mu - (p-2))Y^{p-1}v \\
&= \big((\mu - 2(p-1)) + (p-1)(\mu - (p-2))\big)\, Y^{p-1}v \\
&= \big(\mu p - p^2 + p + 0\big)\, Y^{p-1}v \\
&= p(\mu - p + 1)Y^{p-1}v,
\end{aligned}$$

[8] By the same argument $V_{\lambda+2m} \neq \{0\}$, where $m \in \mathbb{Z}$. Contradiction! What is wrong??

which is what is needed for the induction step.

The result $\langle v, Yv, Y^v, \ldots, Y^n v\rangle = V$ implies that the spaces V_α are one dimensional.

By choosing $p = (\frac{1}{2}(\mu - \lambda) + 1)$ we get $Y^{p-1}v \in V_\lambda$, so $Y^p v = 0$ and

$$0 = X(0) = X(Y^p v) = p(\mu - p + 1)(Y^{p-1}v),$$

from which it follows that $(\mu - p + 1) = 0$ same as $0 = (\mu - \frac{1}{2}(\mu - \lambda) - 1 + 1) = \frac{1}{2}(\mu + \lambda)$, toisin sanoen $\lambda = -\mu$, so V is a sum of one dimensional eigenspaces of H:

$$V = V_{-\mu} \oplus V_{-\mu+2} \oplus \cdots \oplus V_{\mu-2} \oplus V_\mu.$$

In particular, all eigenvalues are even or all odd depending of whether 0 is an eigenvalue or not.

What we have found out, proves that the representation appears in the original list. This is what we wanted to prove.

Terminology remark: The eigenvalues H are called the *weights* of their eigenvectors. The number $\mu \in \mathbb{N}$is the largest weight of the representation in question. [9] □

[9]What is its connection to the dimension of the corresponding symmetric power?