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Undergraduate Representation Theory 2010 **Exercise Set 9**
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space-time coordinates *Tuesday Mar. 16 MaD 302 at LECTURE TIME 14-16*

Problem 1: The Orthogonal Groups. First recall (or assume) the **Implicit Function Theorem:**[1] *Let $f : X \to Y$ be a smooth map of smooth manifolds, and fix a point $y \in Y$. If $d_x f$ is a surjective linear map for all $x \in f^{-1}(y)$, then the subset $X_y = \{x \in X \mid f(x) = y\}$ is a smooth submanifold of X.*

(1) Show that a real $n \times n$ matrix A is in $SO(n)$ if and only if it has determinant one and $AA^t = \mathbb{I}_n$ where $\mathbb{I}_n$ is the $n \times n$ identity matrix.
(2) Find a smooth map $\mathbb{R}^{n^2} \to \mathbb{R}^{\frac{1}{2}n(n+1)}$ such that $SO(n)$ is the pre-image of some point. (Hint: AA^t is symmetric.)
(3) Show that $SO(n)$ is a smooth manifold.
(4) Show that $SO(n)$ is a Lie group.

Problem 2. Lorentz Group

(1) Show that $SO(3,1)$ can be identified with the subgroup of $GL_4(\mathbb{R})$ consisting of matrices A such that

$$A^t \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

(2) Use the implicit function theorem to show that $SO(3,1)$ has a manifold structure.
(3) Show that $SO(3,1)$ is a Lie group. What is its dimension?

Problem 3.

(1) Let V_d be the real vector space of homogeneous polynomials in two variables of degree d. Compute the dimension of this space.
(2) Show that $SL_2(\mathbb{R})$ acts naturally on V_d by "linear change of coordinates,"so that V_d is a representation of $SL_2(\mathbb{R})$.
(3) Is this a smooth representation?
(4) Is it irreducible?

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[1] In calculus classes, this is usually stated with X an open set in $\mathbb{R}^N$ and $Y = \mathbb{R}^d$, in which case the condition that $d_x f$ is surjective becomes the condition that the rank of the linear map $d_x f$ is $N - d$ (which can be stated as the non-vanishing of some $(N-d)$-sized subdeterminant of the Jacobian matrix $\frac{\partial f_i}{\partial x_j}$ at x). The conclusion of the theorem is stated in elementary classes as the existence of an invertible differential function from open sets of $\mathbb{R}^d$ to neighborhoods of each point in X_y, which of course, is the definition of a manifold structure on X_y. The proof in general setting is exactly the same as the proof given in calculus classes, since to check a topological space is a manifold is a local issue in a neigborhood of each point, so it reduces to the case of open sets in Euclidean space.

Problem 4. Some subgroups of $SL_2(\mathbb{R})$

(1) Using the inclusion $SO(2) \subset SL_2(\mathbb{R})$, show that every representation of $SL_2(\mathbb{R})$ induces a representation of $SO(2)$.
(2) Is V_d irreducible as a representation of $SO(2)$? Is it smooth?
(3) Show that $\mathbb{R}^* \times \mathbb{R}^*$ can also be interpreted as a subgroup of $SL_2(\mathbb{R})$, so that the V_d are also smooth representations of $\mathbb{R}^* \times \mathbb{R}^*$.
(4) Is V_d irreducible as a representation of $\mathbb{R}^* \times \mathbb{R}^*$?

Problem 5.

(1) Show that the action of $SL_2(\mathbb{R})$ on the vector space of all 2×2 matrices by left multiplication induces a four dimensional smooth representation of $SL_2(\mathbb{R})$.
(2) Decompose this representation into a sum of two 2-dimensional irreducible representations, each isomorphic to V_1 from Exercise 2. (Hint: row ops!)