JYVÄSKYLÄN YLIOPISTO

MATEMATIIKAN JA
TILASTOTIETEEN LAITOS

Undergraduate Representation Theory 2010 **Exercise Set 8**
author Karen Smith
space-time coordinates *Wednesday Mar. 10 at 8.20-10.00 MaD 380* **!!!**

Problem 1: By definition, a *plane lattice* is a subgroup Λ of $\mathbb{R}^2$ generated by two linearly independent vectors. For example, the lattice generated by $(1,0)$ and $(0,1)$ is the subgroup $\mathbb{Z}^2$ of points in the plane with integer coefficients. The *crystal class* of a lattice Λ is the symmetry group of the lattice[1]—that is, the subgroup of the group of orthogonal transformations $O(\mathbb{R}^2)$ of the plane (with respect to the standard inner product) which preserve the lattice.

(1) Show that all plane lattices are isomorphic *as abstract groups.*
(2) Show that the crystal class of a plane lattice is a finite group.
(3) Find lattices which have crystal classes isomorphic to each of the following groups: $\mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, D_3, D_4, D_6$. (In fact, these are *all* the crystal classes of plane lattices—it is not beyond the reach of an ambitious student to prove this!).
(4) Two lattices are *equivalent* if there is an *orthogonal* change of coordinates of $\mathbb{R}^2$ which transforms one to the other. Show that equivalent lattices have isomorphic crystal classes.
(5) A space lattice and its crystal class are defined analogously. How many distinct space crystal classes can you find?

Problem 2. Rephrase the definition of a topology on a space X in terms of the *closed* sets (meaning the compliments of open sets).

Problem 3: Zariski Topology. For any collection of polynomials $\{f_\lambda\}_{\{\lambda\in\Lambda\}}$ in n-variables with real coefficients, define a set $\mathbb{V}(\{f_\lambda\})$ to be the set of points p in $\mathbb{R}^n$ satisfying $f_\lambda(p) = 0$ for all $\lambda \in \Lambda$.

(1) Show that $\mathbb{R}^n$ has the structure of a topological space whose closed sets are the $\mathbb{V}(\{f_\lambda\})$.
(2) Show that any two non-empty open sets (in the Z-top) have non-empty intersection.
(3) Prove that there is a no self-bjiection of $\mathbb{R}^n$ which transforms the Zariski topology into the Euclidean topology.
(4) Explain how to define a Zariski topology on the set k^n where k is *any field.*
(5) Prove that the Zariski topology on k^n is discrete if and only if k is finite.

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[1]Since the lattice is defined as a *group*, these transformations must preserve the group structure—in particular, they must take the origin to the origin—we don't allow affine shifts of the lattice.

Problem 4: The general linear group GL_n.

(1) Prove that the multiplication map $GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ is a smooth map between open subsets of Euclidean space.
(2) Prove that the inverse map $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ sending g to g^{-1} is a smooth map between open subsets of Euclidean space.
(3) Prove that the multiplication map $GL_n(\mathbb{C}) \times GL_n(\mathbb{C}) \to GL_n(\mathbb{C})$ is a holomorphic map between open subsets of complex Euclidean space. Prove that the inverse map $GL_n(\mathbb{C}) \to GL_n(\mathbb{C})$ sending g to g^{-1} is a holomorphic map between open subsets of complex Euclidean space.
(4) Conclude that $GL_n(\mathbb{R})$ (respectively $GL_n(\mathbb{C})$) is a real (respectively, complex) Lie group. What is its dimension in each case?

Problem 5: Lorentz group. Fix a non-degenerate symmetric bilinear form Q on an n-dimensional real vector space V.

(1) Show that the linear transformations $\phi \in GL(V)$ that respect Q (meaning $Q(\phi(v), \phi(w)) = Q(v, w)$ for all vectors v, w in V) form a subgroup of $GL(V)$. We denote this group $SO_Q(V)$.
(2) Show that if Q is positive definite, then $SO_Q(V)$ is isomorphic to the group $SO_n(\mathbb{R})$ of $n \times n$ real matrices whose columns are orthonormal. (This is the case where the signature of Q is $(n, 0)$).
(3) Show that the groups $SO_Q(V)$ and $SO_{Q'}(V)$ are isomorphic if the signatures of Q and Q' are the same. We use the notation $SO(k, l)$) to denote the isomorphism class of groups $SO_Q(V)$ where the signature of Q is (k, l). The Lorentz group is the special case where the signature is $(3, 1)$.
(4) Prove that $SO(3, 1)$ is a Lie group. What is its dimension? It is not much harder to prove that $SO(k, l)$ is a Lie group in general. What is its dimension?