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Undergraduate Representation Theory 2010 **Exercise Set 7**
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space-time coordinates *Wednesday Mar. 3 at 8.20-10.00 MaD 380* **!!!**

Problem 1: Let V be a finite dimensional *real* representation of a finite group G.

(1) Show that there is a naturally associated complex representation of G, $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$.
(2) Show that V is irreducible if $V_{\mathbb{C}}$ is irreducible.
(3) Give an example to show that V can be irreducible when $V_{\mathbb{C}}$ is not.

Problem 2: Representation of S_4.

(1) Show that there are exactly five conjugacy classes in S_4.
(2) Compute the characters of the trivial E, alternating A and standard W representations of S_4.
(3) Prove that E, A and W are irreducible over $\mathbb{C}$ (hence over $\mathbb{R}$ by Exercise 2).
(4) Prove that there are precisely five irreducible complex representations of S_4 of dimensions $3, 3, 2, 1, 1$.
(5) Prove that $A \otimes W$ is an irreducible representation of S_4, not isomorphic to any of the three in (3).
(6) Complete the character table for S_4.

Problem 3: Another use of the word "character"in mathematics. Unfortunately, the word "character"of a group has a competing meaning used by some authors:

Definition: A (complex) character* of a group G is any group homomorphism $G \to \mathbb{C}^*$

(1) Show that a character* of G is precisely what we have called a one dimensional representation of G.
(2) Give an example to show that characters (over $\mathbb{C}$), as we have defined in class, need not be character*s.
(3) Show that if G is abelian, then the two definitions of character are equivalent.

Problem 4: Group maps in terms of generators and relations Let G be a group, generated by $\{g_1, \dots, g_t\}$.

(1) Show that a group homomorphism $\phi : G \to H$ is completely determined by the images of the generators.
(2) Show that there is a well-defined group homomorphism $\phi : G \to H$ satisfying $\phi(g_i) = h_i$ if and only if for every relation on the g_i in G, the corresponding relation holds for the h_i. That is, more precisely, $\phi(g_i) = h_i$ gives a group homomorphism if and only if whenever $g_{i_1} g_{i_2} \cdots g_{i_t} = e_G$ in G, then the corresponding word $h_{i_1} g_{i_2} \cdots h_{i_t} = e_H$ in H.

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Problem 5: Representations of D_4 Consider the complex representations of D_4.

(1) Using character theory, show that the tautological representation is irreducible over $\mathbb{C}$. Why doesn't the argument we gave over $\mathbb{R}$ hold here?
(2) Show that there are exactly five irreducible representations of D_4, of dimensions $2, 1, 1, 1, 1$.
(3) Explicitly describe the four one dimensional representations of D_4.
(4) Write out the character table of D_4.
(5) Can you make any conclusions about the irreducible real representations of D_4 based on this?

Problem 6: Fix a finite group G. Consider the vector space $\mathcal{F}_G$ of all $\mathbb{C}$-valued functions on G, and the subspace $\mathcal{C}$ of those that are constant on conjugacy classes. We wish to show that the characters of irreducible representations of G span $\mathcal{C}$.

(1) Show that $\alpha \in \mathcal{F}_G$ is constant of conjugacy classes if and only if the map
$$\phi_{\alpha,V} : V \to V; \;\; v \mapsto \sum_{g \in G} \alpha(g) g \cdot v$$
is G-linear for all complex representations V.
(2) Show that the trace of $\phi_{\alpha,V}$ is (α, χ_{V^*}) for all $\alpha \in \mathcal{F}$.
(3) Show if $(\alpha, \chi_{V^*}) = 0$ for some irreducible representation V and $\alpha \in \mathcal{C}$, then $\phi_{\alpha,V}$ is the zero map.
(4) Show that if $\alpha \in \mathcal{C}$ is non-zero, then $\phi_{\alpha,R}$ is not zero, where R is the regular representation.
(5) Conclude that the characters of irreducible representations span $\mathcal{C}$.