Undergraduate Representation Theory 2010 **Exercise Set 6**
author Karen Smith
space-time coordinates *Wednesday Feb. 24 at 14-16 in MaD 302 =Lecture time* **!!!**
Reading: Fulton and Harris pp 3-8, Serre pp 3-7, or Notes.

Problem 1: Consider the action of $\mathbb{Z}_n$ on $\mathbb{R}^2$ defined by letting a generator act by rotation through $2\pi/n$.

(1) Find an explicit group map $\mathbb{Z}_n \to GL_2(\mathbb{R})$ corresponding to this representation. Describe an explicit decomposition into irreducible subrepresentations.
(2) Now consider the "same"action on $\mathbb{C}^2$ given by considering the target of the map described in 1 to be the larger group $GL_2(\mathbb{C})$. (The technical term for this representation is the tensor product of the original real representation with the complex numbers). Describe an explicit decomposition into irreducible sub-representations.

Problem 2: The alternating representation of S_n. Recall that every permutation can be written as a composition of transpositions. Although there are many ways to do this, it is not hard to show that the parity of the number of transpositions is unique. That is, there is a well-defined group homomorphism from S_n to the "even-odd"group ($\mathbb{Z}_2$) sending a permutation to "even"if it is a composition of an even number of transpositions, and to "odd"if it is a composition of an odd number of transpositions.

(1) Let $\sigma \in S_n$ act on $\mathbb{C}$ by multiplication by -1 if σ is odd. Show that this defines a non-trivial one-dimensional representation of S_n. Is it faithful? What is its kernel?
(2) Now let S_n act on $\mathbb{C}^2$ as follows: σ fixes (x, y) if σ is even and σ switches x and y if σ is odd. Prove that this defines a representation of S_n. Is it faithful? What is its kernel? Is it irreducible?
(3) Decompose the representation in (2) completely into irreducible representations.

Problem 3. The **standard representation of** S_3 is the two dimensional subrepresentation of the tautological permutation representation of S_3 on $\mathbb{C}^3$ consisting of the subspace whose coordinates sum to zero.

(1) Let $v \in W$ be an eigenvector for the action of $(123) \in S_3$ for the standard representation W. Prove that corresponding eigenvalue ω of v is a primitive third-root of unity $\omega = e^{2\pi i/3}$ or $e^{4\pi i/3}$.
(2) Let $w = (12) \cdot v$. Prove that w is an eigenvector for the action of (123) with eigenvalue ω^2, and that W is spanned by v and w. (Hint: use a relation in S_3 relating (123) and (12).)
(3) Explicitly describe the action of each element of S_3 in terms of the basis $\{v, w\}$ for W.

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Problem 4. Classification of representations of S_3. Show that if V is any representation of S_3 for which there exist vectors α and β satisfying

(1) α is an eigenvector for (123),
(2) $(12) \cdot \alpha = \beta$,

then α and β span a sub-representation of V. Use this to prove that there are, up to isomorphism, precisely three irreducible representations of S_3 over the complex numbers.

Problem 5: Let V be a finite dimensional representation of a finite group G (over, say $\mathbb{C}$). Define the *character* of this representation to be the function:

$$\chi_V : G \to \mathbb{C}$$
$$g \mapsto \text{trace } g.$$

Show that

(1) The character is constant on conjugacy classes of G: $\chi_V(hgh^{-1}) = \chi(g)$.
(2) $\chi_{V \oplus W} = \chi_V \oplus \chi_W$
(3) $\chi_{V \otimes W} = \chi_V \cdot \chi_W$
(4) $\chi_{V^*} = \frac{1}{\chi_V}$. In particular, over $\mathbb{C}$, $\chi_{V^*} = \overline{\chi_V}$, the complex conjugate.

Problem 6: Compute the characters of the following representations.

(1) The trivial representation of $\mathbb{Z}_4$ on $\mathbb{F}^6$.
(2) The tautological action of D_3 on $\mathbb{R}^2$.
(3) The permutation action of S_3 on $\mathbb{R}^3$.
(4) The standard representation of S_3.
(5) The tautological action of D_4 on $\mathbb{R}^2$.
(6) The vertex permutation action of D_4 on $\mathbb{R}^4$.
(7) The alternating representation of S_3.