Exercise set 1 **Number Theory**
Tuesday SEP 20 2011 at 4-6 pm. in MaD-302

1. *Present the following in base 10*
   (a) 10011_2,
   (b) 1203_4,
   (c) $A0C_{16}$.

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2. *Calculate*
   (a) $1110_2 + 101_2$,
   (b) $230_4 - 101_2$, *give the answer in base 2*
   (c) $32_4 \cdot 23_4$.

3. *Assume* $k \in \mathbb{N}$, $k > 1$.
   (a) *Find the base of the number system* k, *such that* $28 = 124_k$.
   (b) *Calculate* $101_k + 101_{k^2}$.

4. *Prove:*
   Assume $n, m, d \in \mathbb{Z}$.
   (a) *If* $d \mid n$ *and* $n \mid m$, *then* $d \mid m$.
   (b) *If* $d \mid n$ *and* $d \mid m$, *then* $d \mid (an + bm)$ *for all* $a, b \in \mathbb{Z}$.
   and prove by induction:
   (c) *Let* $m \in \mathbb{Z} \setminus \{0\}$ *and* $n \in \mathbb{N}$. *If* $a_i \in \mathbb{Z}$ *and* $m \mid a_i$ *for all* $i = 1, 2, \ldots n$, *then*
   $$m \mid (c_1 a_1 + c_2 a_2 + \cdots + c_n a_n)$$
   for all $c_i \in \mathbb{Z}$, $i = 1, 2, \ldots, n$.

5. *Olkoot* $m \in \mathbb{Z} \setminus \{0\}$ *ja* $n \in \mathbb{N}$, $n \geq 2$. *Näytä, että jos* $a_i \in \mathbb{Z}$, $m \mid a_i$ *kaikilla* $i = 1, 2 \ldots n - 1$ *ja* $m \nmid a_n$, *niin*
$$m \nmid (a_1 + a_2 + \cdots + a_n).$$

6. *Which of the follwing are true? Proof or counterexample.*
   (a) *If the number* $k \in \mathbb{Z}$ *is divisible by* 5 , *then* $(k + 5)^{10}$ *is divisible by* 5.
   (b) *Let* $a, b, c, d \in \mathbb{Z}$, $a \mid b$ *and* $c \mid d$. *Then* $(a + c) \mid (b + d)$.
   (c) *Fir natural numbers* a, b, n *with* $a^2 \mid n$, $b^2 \mid n$ *and* $a^2 \leq b^2$ *one always has* $a \mid b$.

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For fxercises (7) and (8) use the **Euclidean algorithm**:

(a, b) is found like this (if $a \geq b > 0$). Define the numbers r_j by $r_0 = a,\ r_1 = b$ and generally by the division algorithm:

$$r_{j-2} = r_{j-1}q_{j-1} + r_j, \quad 0 \leq r_j < r_{j-1}, j = 2, 3, \ldots, n+1.$$

The first 3 equations are

$$a = bq_1 + r_2,$$
$$b = r_2 q_2 + r_3.$$
$$r_2 = r_3 q_3 + r_4.$$

Since generally $(a, b) = (a + kb, b)$,

$$d = (a, b) = (a - bq_1, b) = (r_2, b), \text{ same as } (r_0, r_1) = (r_1, r_2).$$

Continue the same way, and arrive at

$$d = (r_j, r_{j+1}), \quad \forall\ j = 0, 1, \ldots, n.$$

But $r_{n+1} = 0$,so $(r_n, r_{n+1}) = r_n$. All in all

$$d = r_n,$$

Finally x, y in $d = ax + by$ can be found by reversing the calculation.

Example

Find (252,198):

$252 = 1 \cdot 198 + 54$

$198 = 3 \cdot 54 + 36$

$54 = 1 \cdot 36 + 18$

$36 = 2 \cdot 18$

Siis $(252, 198) = 18 = 54 - 36 = \ldots = 4 \cdot 252 - 5 \cdot 198$.

7. *Calculate* $(1492, 1066)$ *using Euclid'd algorithm.*

8. *Find* $x, y \in \mathbb{Z}$, *s. th.*

$$(1492, 1066) = 1492x + 1066y.$$

9. *Find numbers* $a, b, c \in \mathbb{Z}$ *s.th.*

(1) $a \,|\, c$ *and* $b \,|\, c$ *but* $ab \not| \, c$,

(2) $a \,|\, bc$ *but* $a \not| \, c$.